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STOCHASTIC ANALYSIS OF A DEMOGRAPHIC MODEL OF URBANIZATION

G.P. SAMANTA
(Received 27.08.2002)

ABSTRACT

The continuous time version of Rogers' demographic model is considered. The demographic parameters describing the interactions are assumed to be represented by a dichotomic Markov process. Explicit expressions are derived for the time development of the average of the urban and rural population size and their asymptotic behaviours are discussed.

Mathmatics subject classification (1980) : 92A17

Keywords : Demographic model, Dichotomic markov process, Urbanization, Stability, Delta-correlated Stochastic process.

INTRODUCTION

The study of long term growth and development of urbanization is essential for optimum provision of adequate and efficient public services in the urban sector and for the formulation and evaluation of public policy. Hence, the need for developing quantitative models of urbanization for improved population predictions. It is expected that the analysis of such models would facilitate the comprehension of the underlying mechanism of the phenomenon of urbanization. There have been several attempts to formulate demographic models for urban growth; important among them are the quantitative models by Keyfitz [1], Rogers [2] and the United Nations [3]. These three demographic models are discussed in detail in an article by Ledent [4].

The continuous time version of Rogers' demographic model (which was originally for discrete time) can be expressed by a pair of coupled linear differential equations [4]. Denoting by U (t) the urban population and by R (t) the rural population, this model becomes

$$\frac{dR}{dt} = (\alpha - m)R + \gamma U \tag{1a}$$

(Department of Mathematics Bengal Engineering College (D.U.), Howrah - 711 103, India).

$$\frac{dU}{dt} = (\beta - \gamma)U + mR \tag{1b}$$

where α is the natural growth rate of the rural population, β is that of the urban population, m is the in-migration rate per individual of rural population (from rural to urban sector) and γ is the out-migration rate per urbanite (from urban to rural area).

In the present paper it is proposed to study the mean behaviour of the rural and urban populaton size for the demographic model described by equations (1) with demographic parameters stochasticized in order to take account of the effect of fluctuating environment. If we consider a physical situation in which the random environmental fluctuations do not occur with great rapidity in comparison with the time-scale of population growth, it is natural to assume that the stochastic parameters have finite correlation time.

MODIFIED STOCHASTIC DEMOGRAPHIC MODEL

The modified stochastic version of equations (1) to be considered is

$$\frac{d}{dt} X_1(t) = \beta_1(t) X_2(t) + \gamma_1(t) X_1(t) \tag{2a}$$

$$\frac{d}{dt} X_2(t) = \beta_2(t) X_1(t) + \gamma_2(t) X_2(t) \tag{2b}$$

where $X_1(t) = R(t), X_2(t) = U(t), \beta_1(t) = \gamma(t), \beta_2(t) = m(t)$,

$\gamma_1(t) = \alpha(t) - m(t)$ and $\gamma_2(t) = \beta(t) - \gamma(t)$.

The stochastic parameters $\beta_i(t)$ and $\gamma_i(t)$ can be expressed as

$$\beta_i(t) = \beta_{io}[1 + \epsilon_i \, \Delta(t)], i = 1,2 \tag{2c}$$

$$\gamma_i(t) = \gamma_{io}[1 + \epsilon_{i+2} \, \Delta(t)], i = 1,2 \tag{2d}$$

where β_{io} and γ_{io} are constants and the ϵ_i's are also constants but positive. $\Delta(t)$ is assumed to be a dichotomic Markov process (DMP) [5] defined as a two valued stepwise constant Markov process with equiprobable values +1 and -1 and transition probabilities (for $t_2 > t_1$) :

$$P\{\Delta(t_2) = +1|\Delta(t_1) = +1\} = P\{\Delta(t_1) = -1\Delta(t_1) = -1\}\}$$

$$= \frac{1}{2}[1 + e^{-\nu(t_2 - t_1)}]$$

$$P\{\Delta(t_2) = +1|\Delta(t_1) = -1\} = P\{\Delta(t_2) = -1|\Delta(t_1) = +1\}$$

$$= \frac{1}{2}[1 - e^{-\nu(t_2 - t_1)}]$$

Here $1/\nu$ is the correlation time.

For a DMP, we have

$E[\Delta(t)] = 0, E[\Delta^2(t)] = 1,$

$$E[\Delta(t_1)\Delta(t_2)] = e^{-\nu|t_1 - t_2|} \tag{3}$$

Also the closure property holds exactly, i.e. if $\psi\{\Delta(.)\}$ is a functional of the process involving only times prior to t_1, then

$$E[\Delta(t_1)\Delta(t_2)\Psi\{\Delta(.)\}] = E[\Delta(t_1)\Delta(t_2)]E[\Psi\{\Delta(.)\}] \tag{4}$$

AVERAGES OF THE POPULATIONS

Equations (2) can be written in matrix form:

$$\frac{d}{dt}X(t) = AX(t) + \Delta(t)BX(t)$$

where $X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}$, $A = \begin{bmatrix} \gamma_{10} & \beta_{10} \\ \beta_{10} & \gamma_{20} \end{bmatrix}$, $B = \begin{bmatrix} \epsilon_3 \gamma_{10} & \epsilon_1 \beta_{10} \\ \epsilon_2 \beta_{20} & \epsilon_2 \gamma_{20} \end{bmatrix}$ (5)

This has the solution

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-t')}\Delta(t')BX(t')dt'$$

Iterating,

$$X(t)=e^{At}X(0)+\int_0^t dt'\, e^{A(t-t')}\Delta(t')B[e^{At'}X(0)+\int_0^{t'} e^{A(t'-t'')}\Delta(t'')BX(t'')dt'']$$

Taking the ensemble averages, we get,

$$E[X(t)]=e^{At}X(0)+\int_0^t dt'\, e^{A(t-t')}B\{\int_0^{t'} E[\Delta(t')\Delta(t'')e^{A(t'-t'')}BX(t'')dt''] \tag{6}$$

Here we have used the result E $[(\Delta(t)]=0$

Therefore $E[X(t)]=e^{At}X(0)+\int_0^t e^{A(t-t')B\varnothing(t')dt'}$ (7)

where $\varnothing$ (t') = $\int_0^{t'} e^{(A-\nu I)(t'-t'')}BE[X(t'')]dt''$ (8)

Thaking Laplace transform, (7) and (8) yield respectively

$$E\left[\overline{X}(s)\right]=[sI-A]^{-1}[X(0)+B\overline{\varnothing}(s)]$$

and $\overline{\varnothing}(s)=\left[(s+\nu)I-A\right]^{-1}BE\left[\overline{X}(s)\right]$

whence $\left[sI-A-B[(s+\nu)I-A]^{-1}B\right\}E[\overline{X}(s)]=X(0)$ (9)

Now writing D = $(s+\nu-\gamma_{20})(s+\nu-\gamma_{10})-\beta_{10}\beta_{20}$, it is seen that

$$[(s+\nu)I-A]^{-1}=\frac{1}{D}\begin{bmatrix} s+\nu-\gamma_{20} & \beta_{10} \\ \beta_{20} & s+\nu-\gamma_{10} \end{bmatrix}$$

Using the value of B from (5), we get

$$B[s+\nu]I-A]^{-1}B=\frac{1}{D}\begin{bmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{bmatrix},$$

where

$$\mu_{11}=\epsilon_3^2\,\gamma_{10}^2(s+\nu-\gamma_{20})+\epsilon_1\epsilon_3\,\beta_{10}\beta_{20}\gamma_{10}+\epsilon_2\epsilon_3\,\beta_{10}\beta_{20}\gamma_{10}+\epsilon_1\epsilon_2\,\beta_{10}\beta_{20}(s+\nu-\gamma_{10}),$$

$$\mu_{12} = \epsilon_1 \epsilon_3 \beta_{10}\gamma_{10}(s+\nu-\gamma_{20}) + \epsilon_1^2 \beta_{10}^2\beta_{20} + \epsilon_3 \epsilon_4 \beta_{10}\gamma_{10}\gamma_{20} + \epsilon_1 \epsilon_4 \beta_{10}\gamma_{20}(s+\nu-\gamma_{10}),$$

$$\mu_{21} = \epsilon_2 \epsilon_3 \beta_{20}\gamma_{10}(s+\nu-\gamma_{20}) + \epsilon_3 \epsilon_4 \beta_{20}\gamma_{10}\gamma_{20} + \epsilon_2^2 \beta_{10}\beta_{20}^2 + \epsilon_2 \epsilon_4 \beta_{20}\gamma_{20}(s+\nu-\gamma_{10}),$$

$$\mu_{22} = \epsilon_1 \epsilon_2 \beta_{10}\beta_{20}(s+\nu-\gamma_{20}) + \epsilon_1 \epsilon_4 \beta_{10}\beta_{20}\gamma_{20} + \epsilon_2 \epsilon_4 \beta_{10}\beta_{20}\gamma_{20} + \epsilon_4^2 \gamma_{20}^2(s+\nu-\gamma_{10}),$$

Writing $F = \left(s-\gamma_{10}-\frac{\mu_{11}}{D}\right)\left(s-\gamma_{20}-\frac{\mu_{22}}{D}\right)-\left(\beta_{10}+\frac{\mu_{12}}{D}\right)\left(\beta_{20}+\frac{\mu_{21}}{D}\right),$

we find that

$$\{sI - A - B[(s+\nu)I - A]^{-1}B\}^{-1} = F^{-1}\begin{bmatrix} s-\gamma_{20}-\frac{\mu_{22}}{D} & \beta_{10}+\frac{\mu_{12}}{D} \\ \beta_{20}+\frac{\mu_{21}}{D} & s-\gamma_{10}-\frac{\mu_{11}}{D} \end{bmatrix} \tag{10}$$

From (9) and (10), we have

$$E\left[\overline{X}_1(s)\right] = F^{-1}[\{s-\gamma_{20}-\frac{\mu_{22}}{D}\}X_1(0) + \{\beta_{10}+\frac{\mu_{12}}{D}\}X_2(0)] \tag{11a}$$

and $$E\left[\overline{X}_2(s)\right] = F^{-1}[\{s-\gamma_{10}-\frac{\mu_{11}}{D}\}X_2(0) + \{\beta_{20}+\frac{\mu_{21}}{D}\}X_1(0)] \tag{11b}$$

$E[X_1(t)]$ and $E[X_2(t)]$ are respectively the inverse Laplace transform of (11), and their evaluation is laborious. Equations (11) give the values of $E[\overline{X}_1(s)]$ and $E[\overline{X}_2(s)]$ for different magnitudes of stochastic parametric perturbations.

DISCUSSION OF THE STABILITY OF A DELTA-CORRELATED STOCHASTIC PROCESS

Here we consider a physical situation in which the environmental changes occur with great rapidity as compared to the time-scale of population growth and consequently we take the random fluctuations due to them to be delta-correlated and the corresponding results can be obtained in the limit of infinitesimally short correlation time. To this end we let $\epsilon_i, \nu \to \infty$ such that $\left(\epsilon_i^2/\nu\right) \to c_i$, i = 1, 2, 3, 4. Then the autocorrelation function

$$E[\epsilon_i \, \Delta(t_1) \, \epsilon_i \, \Delta(t_2)] \to 2c_i \delta(t_1 - t_2).$$

Also $\dfrac{\mu_{11}}{D} \to c_3\gamma_{10}^2 + \beta_{10}\beta_{20}\sqrt{c_1c_2}$,

$$\frac{\mu_{12}}{D} \to \beta_{10}\sqrt{c_1}[\gamma_{10}\sqrt{c_3} + \gamma_{20}\sqrt{c_4}],$$

$$\frac{\mu_{21}}{D} \to \beta_{20}\sqrt{c_2}[\gamma_{10}\sqrt{c_3} + \gamma_{20}\sqrt{c_4}],$$

$$\frac{\mu_{22}}{D} \to c_4\gamma_{20}^2 + \beta_{10}\beta_{20}\sqrt{c_1c_2}.$$

Thus for a delta-correlated process,

$$E[\overline{X}_1(s) = G^{-1}[\{s - \gamma_{20} - c_4\gamma_{20}^2 - \beta_{10}\beta_{20}\sqrt{c_1c_2}\}X_1(0) + \beta_{10}\{1 + \gamma_{10}\sqrt{c_1c_3} + \gamma_{20}\sqrt{c_1c_4}\}X_2(0) \tag{12a}$$

$$E[\overline{X}_{20}(s) = G^{-1}[\beta_{20}\{1 + \gamma_{10}\sqrt{c_2c_3} + \gamma_{20}\sqrt{c_2c_4}\}X_1(0) + \{s - \gamma_{10} - c_3\gamma_{10}^2 - \beta_{10}\beta_{20}\sqrt{c_1c_2}\}X_2(0)] \tag{12b}$$

where

$$G = \{s - \gamma_{10} - c_3\gamma_{10}^2 - \beta_{10}\beta_{20}\sqrt{c_1c_2}\}\{s - \gamma_{20} - c_4\gamma_{20}^2 - \beta_{10}\beta_{20}\sqrt{c_1c_2}\}$$
$$- \beta_{10}\beta_{20}\{1 + \gamma_{10}\sqrt{c_1c_3} + \gamma_{20}\sqrt{c_1c_4}\}\{1 + \gamma_{10}\sqrt{c_2c_3} + \gamma_{20}\sqrt{c_2c_4}\}$$

(12) can be written as

$$E\left[\overline{X}_1(s)\right] = \frac{(s+b)X_1(0) + cX_2(0)}{(s+a)(s+b)+d} \tag{13a}$$

and $$E\left[\overline{X}_2(s)\right] = \frac{(s+a)X_2(0) + c'X_1(0)}{(s+a)(s+b)+d} \tag{13b}$$

where

$$a = -\left(\gamma_{10} + c_3\gamma_{10}^2 + \beta_{10}\beta_{20}\sqrt{c_1c_2}\right),$$

$$b = -\left(\gamma_{20} + c_4\gamma_{20}^2 + \beta_{10}\beta_{20}\sqrt{c_1c_2}\right),$$

$$c = \beta_{10}(\gamma_{10}\sqrt{c_1c_3} + \gamma_{20}\sqrt{c_1c_4} + 1),$$

$$c' = \beta_{20}(\gamma_{10}\sqrt{c_2c_3} + \gamma_{20}\sqrt{c_2c_4} + 1),$$

$$d = -cc'.$$

Let $\alpha, \beta = -\frac{1}{2}[-(a+b) \pm \sqrt{(a-b)^2 - 4d}]$

Taking the inverse Laplace transform of (13),

$$E[X_1(t)] = \frac{1}{\alpha - \beta}[\{(\alpha + b)X_1(0) + cX_2(0)\}e^{\alpha t} - \{(\beta + b)X_1(0) + cX_2(0)\}e^{\beta t} \quad (14a)$$

$$E[X_2(t)] = \frac{1}{\alpha - \beta}[\{(\alpha + a)X_2(0) + c'X_1(0)\}e^{\alpha t} - \{(\beta + a)X_2(0) + c'X_1(0)\}e^{\beta t}] \quad (14b)$$

Let us assume that the stochastic perturbation terms are the same. Hence $c_1 = c_2 = c_3 = c_4.$ Now, $K^2 = (a-b)^2 - 4d = \left[1 + c_1(\gamma_{20} + \gamma_{10})\right]^2\left\{(\gamma_{20} - \gamma_{10})^2 + 4\beta_{10}\beta_{20}\right\} > 0$ (15)

For asymptotically large times, $E[X_1(t)]$ and $E[X_2(t)]$ are unstable for $-(a+b) + K > 0$, i.e. $\gamma_{10} + \gamma_{20} > -c_1\left(\gamma_{10}^2 + \gamma_{20}^2 + 2\beta_{10}\beta_{20}\right) - K$ (16)

If $\gamma_{10} + \gamma_{20} < -c_1\left(\gamma_{10}^2 + \gamma_{20}^2 + 2\beta_{10}\beta_{20}\right) - K$, then $E[X_1(t)]$, $E[X_2(t)]$, will tend to zero, i.e. they will extinct after a long time span.

For $\gamma_{10} + \gamma_{20} = -c_1\left(\gamma_{10}^2 + \gamma_{20}^2 + 2\beta_{10}\beta_{10}\right) - K,$ (17)

$$\lim_{t\to\infty} E[X_1(t)] = \frac{1}{K}\{(\alpha + b)X_1(0) + cX_2(0) \quad (18a)$$

$$\lim_{t\to\infty} E[X_2(t)] = \frac{1}{K}\{(\alpha + b)X_2(0) + c'X_1(0)\} \quad (18b)$$

In this case, we have

$$\lim_{t\to\infty}\frac{E[X_1(t)]}{E\left[X_1(t)+E\left[X_2(t)\right]\right.} = \frac{(\alpha+b)X_1(0)+c'X_2(0)}{\Delta} \tag{19a}$$

and

$$\lim_{t\to\infty}\frac{E[X_2(t)]}{E\left[X_1(t)+E\left[X_2(t)\right]\right.} = \frac{(\alpha+a)X_2(0)+c'X_1(0)}{\Delta} \tag{19b}$$

where $\Delta=(\alpha+a+c)X_2(0)+(\alpha+b+c')X_1(0)$

After some simple calculations, we have

$$\alpha+a=\left\{1+c_1(\gamma_{20}+\gamma_{10})\right\}\left[\left(\frac{1}{2}(\gamma_{20}-\gamma_{10})+\left\{\frac{1}{4}(\gamma_{20}-\gamma_{10})^2+\beta_{10}\beta_{20}\right\}^{1/2}\right]\right. \tag{20}$$

$$\alpha+b=\left\{1+c_1(\gamma_{20}+\gamma_{10})\right\}\left[\left(\frac{1}{2}(\gamma_{10}-\gamma_{20})+\left\{\frac{1}{4}(\gamma_{20}-\gamma_{10})^2+\beta_{10}\beta_{20}\right\}^{1/2}\right]\right. \tag{21}$$

If $\gamma_{20}>\gamma_{10},\beta_{20}\geq\beta_{10}$ or $\gamma_{20}\geq\gamma_{10},\beta_{20}>\beta_{10}$, then $\alpha+a>c$ and $\alpha+b>c'$,

therefore $\dfrac{(\alpha+b)X_1(0)+cX_2(0)}{\Delta}<\dfrac{(\alpha+a)X_2(0)+c'X_1(0)}{\Delta}$

Therefore, we conclude that if mean value of $(\beta-\gamma)>$ mean value of $(\alpha-m)$ and mean value of m $\geq$ mean of γ or mean value of $(\beta-\gamma)\geq$ mean value of $(\alpha-m)$ and mean value of m > value of γ, then we have a high level of urbanization after a long time span.

If mean value of $(\beta-\gamma)$ = mean value of $(\alpha-m)$ and mean value of m = mean value of γ, then the level of urban population and the level of rural population will be equal after a long time span.

It may be noted that for $\beta_{10}=0$ (no migration from urban areas) Roger's model reduces to Keyfitz's model. For this model if the condition (17) holds good and mean value of $\beta-\gamma\geq$ mean value of $(\alpha-m)$, then

$$\lim_{t\to\infty}\left[\frac{E[X_1(t)]}{E[X_1(t)]+E[X_2(t)]}\right]=0 \tag{22a}$$

$$\lim_{t\to\infty}\left[\frac{E[X_2(t)]}{E[X_1(t)]+E[X_2(t)]}\right]=1 \tag{22b}$$

Therefore, in this case the entire population will eventually be urbanized.

CONCLUDING REMARKS

We have carried out an asymptotic analysis of a continuous stochastic version of Rogers' demographic model. It is seen that under some conditions this model holds up high level of urbanization. From this continuous stochastic version of Rogers' demographic model we have also duduced a continuous stochastic version of Keyfitz's demographic model here we have seen that under some conditions the entire population will eventually be urbanized.

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CALENDAR SYSTEMS IN ANCIENT INDIA

G. S. PANDEY*
(Received 09-03-2003)

ABSTRACT

Ever since the dawn of Vedic civilization in India the counting of time was considered as the highest need of the society. During the Ṛgvedic period a solar year was recognized as formed of twelve months and each month of 30 days. Vedānga Jyotiṣa (1400 B.C.) introduced the concept of Yuga system of five saṃvatsaras(years) and made a clear distinction between a tithi (lunar day), Sāvana day and solar day. Also, in order to account for a solar year, the system of intercalary months was introduced.

In this exposition we, present a short account of the main fifteen calendar systems introduced in India from time to time. In particular, we trace the history of Vikram Saṃvata (initiated from 57 B.C.) formed on the basis of Surya Siddhānta, which is the only calendar system being used, with some variations, almost everywhere in India, while the Government of India has declared an alien, Saka Saṃvata (started from 78 A.D.), as the national calender.

Mathematics Subject Classifications : 01 A 32, 85-03, 01 A 10.

Key Words & Phrases : Ṛigveda, Vedānga Jyotiṣa, Yuga, Tithi, Sāvana day, Solar day, Intercalary month, Kaliyuga, Surya Siddhānta, Vikrama Saṃvata and Saka Saṃvata.

INTRODUCTION

A calendar system is the first need of a civilized society. This is the reason that Ācārya Varāḥmihira emphatically writes [Var b, Chapter II,8]:

"अप्रदीपा यथा रात्रिर् ऽनादित्यम् यथा नभः।
तथा असंवत्सरोराजा भ्रमन्ति ऽन्धइवध्वनिः।।"

That is, a king without calendar system is like a night without a lamp, the sky without the Sun and a blind man moving on the sound.

Ancient Indian scholars were keen observers of the celestial bodies and capable to explain a number of phenomena in the sky without any mechanical or optical instruments. The rise and setting of the Sun and Moon were carefully

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recorded for the formulation of a suitable calendar. Also, it was recognized that the light of the Sun was composed of seven rays as in the following hymn of Ṛgveda [Rig, I, 16, 14, 2]:

"सप्त युज्जन्ति रथमेक चक्रमेको अश्वो वहति सप्तनाम।"

That is, the one-wheel chariot of (the Sun) is drawn by one horse bearing seven names.

During the Ṛgvedic period it was estimated that one year consisted of twelve months composed of three hundred sixty days and three hundred sixty nights as in the following couplet [Rig ; I, 164, 11]:

"द्वादशारं नाहिंताज्जराय वर्षर्ति चक्रं परिद्यामृतस्य।
आपुत्राऽग्ने मिथुना सो ऽत्र सप्त शतानि विंशतिश्च तस्युः।।"

That is, the Sun's wheel with twelve spokes (months) revolves around (the Earth) and is never destroyed. Oh Fire! on this wheel are mounted seven hundred and twenty people (360 days and 360 nights).

Vedic scholars recognized that the seasons were caused by the Sun [Rig ; I 95, 3]:

"पूर्वा मनुप्रादिशं पार्थिवाना मृतून प्राशा सद्विदधावनुष्ठुः।"

That is, controlling the seasons (the Sun) creates the East at various places.

The above hymn indicates that the seasons are caused by the Sun and the eastern direction at every point is ascertained by the position of the Sun in the morning, which also suggests that the Earth is round and the local time differed at different places. Taittirīya Saṃhitā, on the other hand, asserts that the phases of the moon and its light are provided by the Sun [Tat; III, 41, 1]:

"चन्द्रमा ऽमावस्या आदित्य मनु प्रविशति।
सूर्य रश्मिश्चन्द्रमा गन्धर्व।।"

According to the climatic conditions of India, the whole year was divided into six parts and each part was called a ṛitu (ऋतु) as given in the following verse of Taittirīya Saṃhitā (cf.[Tat;4, 11]:

"मधुश्च माधवश्च वासंतिकावृत्। शुक्रश्च शुचिश्च ग्रैष्मावृत।।
नभश्च नभस्यश्च वार्षिकावृत्। इषश्चोर्जश्च शारदा वृत्।।
सहश्च सहस्यश्च हेमन्तिकावृत । तपश्च तपस्यश्च शैशिरावृत्।।"

That is, Madhu and Mādhava form the spring season, Śukra and Śuci the summer, Nabhasa and Nabhaṣya are the months of rainy season, Iṣa and Urja are Sarada, Sahasa and Sahasya form the Hemant, and Tapasa and Tapasya are the Sisira season.

SAMṂVATSARA AND ITS DIVISIONS

Taittirīya Saṃhitā provides the composition of a saṃvatsara in detail as in the following verses [tat ; V, 6, 7] :

"षड् रात्रीर्दीक्षित: स्यात् षड्वा ऋतव: संवत्सर:।
द्वादश रात्रीर्दीक्षित: स्यात् द्वादश मासा: संवत्सर:।।
त्रयोदश रात्रीर्दीक्षित: स्यात् त्रयोदश मासा: संवत्सर:।
पंचदश रात्रीर्दीक्षित: स्यात्पन्चदश वा अर्धमासस्य रात्रयोर्धमासश: संवत्सर आप्यते।।
चतुर्विंशति रात्रीर्दीक्षित: स्याच्चतुर्विशतिरर्ध मासा: संवत्सर:।
त्रिशंत्रात्रीर्दीक्षित: स्यात् त्रिंशदक्षराविराट् - मासं दीक्षित: स्याद्यो मास: स संवत्सर:।।"

That is, be instructed in six nights because Saṃvatsara is of six seasons. Be instructed in twelve nights as saṃvatsara is of twelve months. Be instructed in thirteen nights as saṃvatsara is of thirteen months. Be instructed in fifteen nights as half-month has fifteen nights and half-months form saṃvatsara. Be instructed in twenty four nights, because saṃvatsara has twenty four half-months. Be instructed in thirty nigths, thirty letters form Virat. Be instructed in a month for months form a saṃvatsara.

The above verse affirms that

1 Saṃvatsara = [6 seasons.
12 months = 24 half months.
13 months (including one inteacalary month).
24 half - months.

As pointed out by Dikshit [Dik, pp. 42 - 43], the Vedic scholars regarded a lunar month of less than 30 days and, therefore, to account for a solar year it was necessary to add one intercalary month usually after 30 lunar months.

Although Nakshatras (constellations) are mentioned in Ṛgveda at a number of places, total list of 27 constellations are given in Taittirīya Saṃhīta (cf.(Tat) ; IV, 4/10 and, see also [Tat], 69 - 81, for some other details).

THE CONCEPT OF YUGA

During the Vedic and Vedāṅga periods the concept of five years Yuga was recognized. The year was called Saṃvatsaras and one Yuga consisted of five Saṃvatsaras. The first two verses of Ṛgvedic Jyotiṣa clearly affirms [cf. (Dik), p. 98] that:

"पन्चसंवत्सरमयं युगाध्यक्षं प्रजापतिम्।
दिनर्त्वयनमासाङ्गं प्रणम्य शिरसा शुचिः।। 1 ।।
प्रणम्य शिरसा कालमभिवाद्य सरस्वतीम्।
कालज्ञानं प्रवक्ष्यामि लगधस्य महात्मनः।। 2।।"

That is to purify ourselves we pray to Prajāpati, the creator of Yuga composed of five Saṃvatsaras including days, seasons, ayans and months.

Praying to time and Saraswati (the goddess of learning), we describe the knowledge of time of Logadha.

The name of five saṃvatsaras are given in Taittirīya Brāhmaṇa in the form [Tat, III, 10/4] :

"संवत्सरोसि परिवत्स रोसि।
इदावत्सरोसि दुवत्सरोसि।।
इद्वत्सरोसिवत्सरोसि।........"

The above lines give names of five Saṃvatsaras as Saṃvatsara, Parivatsara, Idavatsara, Iduvatsara and Idvatsara.

Vedāṅga Jyotiṣa provides further extension to Vedic calender system. This classical text on Jyotiṣa contains 45 verses on ancient Indian astronomy. As mentioned above, Vedāṅga Jyotiṣa was composed by Lagadh around the year 1400 B.C. Dikshit (Dik, p. 94) has classified Vedāṅga Jyotiṣa into the following three sections.

(i) Ṛgveda Jyotiṣa.

(ii) Yajurveda Jyotisa.

(iii) Atharvaveda Jyotisa.

Of course, a number of verses of the above Jyotiṣa. texts are similar. Vedāṅga Jyotiṣa provides the divisions and subdivisions of a Yuga in full detail as in the following couplet of Yajurveda Jyotiṣa. (cf. [Dik]; p 119, Yajurjyotiṣa Verse 27) :

"त्रिशत्यह्नां सषट षष्टिरब्द: षडऋतबो ऽयने।
मासा द्वादश सूर्या: स्युरेतत्पन्च गुणं युगम्।।"

That is, a Saṃvatsara consists of 366 days, six seasons, two ayans and twelve solar months. Yuga is its five times. Yajurjyotiṣa [loc. cit; p. 120, Verse 29] further specifies :

"उदयावासवस्य स्युर्दिन राशि: स्वपन्चक:।
ऋषेर्द्धि षष्टिहीनं स्यात् विंशत्या चैकया स्तृणाम्।।"

That is, in a Yuga the number of sunrise is five times the number of days in a solar year, while moon-rise is 62 less than that.

The above couplet asserts that in 1 Yuga the sun rises 5 x 366 = 1830 times, while the number of moon rise is 1830 - 62 = 1768 times. It was known that the duration of time from one moonrise to the next was greater than one solar day. In fact, in one lunar month there can be 29 or 30 sunrise, while moonrise 28 or 29. the definition of a tithi depends on the motion of the Moon. It has been defined

as the time from a moon rise to the next. Savan day (अहोरात्र), on the other hand, is the time from a sunrise to the following, Yajurjyotiṣa makes a clear distinction between a tithi, Savan day and solar day as in the verse [Yajurveda Jyotiṣa, Verse 31]:

"सावनेन्दुस्तृमासानां सष्टि: सैका द्विसप्तिका।
द्युत्रिंशत् सावन: सार्ध: सूर्य: स्तृणां सपर्यय:।।"

(In a Yuga) there are 61 Savan months, 62 luner months and 67 nakshatra months (स्तृमास). A savan month is of 30 days and a solar month of 30 ½ savan days.

Hence it is clear that a Yuga contains 60 solar months, 61 Savan months and 62 lunar months. Since a Yuga contains 1830 days, a Savan month is of 1830 ÷ 61 = 30 days. Also, one solar month = 1830 ÷ 60 = 30 ½ days. The following verse of Yajurjyotiṣa (loc. cit., Verse 37) provides the relationship between a Savan day, a solar day and a tithi and the intrinsic cause for the addition of an intercalary month to a lunar calender system:

"द्यूनं द्विषष्टि भागेन हेयं सूर्यात् सपार्वणम्।
यत्कृतावुपजायेते मध्ये चान्ते चाधि मासकौ।।"

That is, by subtracting 62 nd part from a Savan day a tithi (lunar day) is obtained, while adding up the 60th part to a Savan day it becomes a solar day. Since a Savan days is smaller than a solar day, the intercalary month occurs in the middle of end or a Yuga.

According to the above verse, we see that

$$1 \text{ Tithi} = \frac{1830}{62\text{x}30} \text{ Savan days}$$

$$= 61/62 = 1 - 1/62 \text{ Savan days}$$

and

$$1 \text{ solar day} = 30\ ½ \div 30 \text{ Savan days}$$

$$= 61 / 60 = 1 + 1 / 60 \text{ Savan days.}$$

Hence, in order to account for a solar year, the addition of an intercalary month in a Yuga becomes necessary.

Ṛgveda Jyotiṣa (Ṛgjotisa), on the other hand, asserts that an intercalary month occurs after 30 Savan months as in the verse (cf. [Dik], p. 114) :

"यदर्ध दिन भागानां सदा पर्वणि पर्वणि।
ऋतु शेषं तु तद्विद्यात् संख्याय सहपर्वणाम्।। 23।।"

That is, the sum of half parts of tithis from all Parvas (half lunar months) forms a Rituśeṣa (remainder of lunar - month).

As mentioned above, one Yuga is formed of 1830 Savan days, 120 half - solar months and 124 half-lunar months (Parvas).

Hence we see That

Half lunar - month (Parva) = 1830 / 124 Savan days

$$= 14\frac{94}{124} \quad \text{Savan days,}$$

Half solar-month $$= \frac{1830}{120} \quad \text{Savan days}$$

$$= \frac{61}{4} \quad \text{Savan days}$$

$$= 15\,\tfrac{1}{4} \quad \text{Savan days}$$

$$= 14\frac{94}{124} + \frac{61}{124} \quad \text{Savan days}$$

Thus we have

(Half solar - month) - (Half lunar - month)

$$= 15\tfrac{1}{4} - 14\frac{94}{124}$$

$$= \frac{61}{124} \quad \text{Savan days.}$$

This means, aproximately half Savan days is the remainder. Therefore the difference in 30 Savan months (or 60 half Savan Months) becomes

$$\frac{60 \times 61}{124} = 29\frac{64}{124} \quad \text{Savan days}$$

$$= 1 \text{ lunar - month.}$$

Hence, according to Rgjjotiṣa, after 30 Savan months 1 lunar - months has to be added to account for solar years.

As mentioned above, the Savan days form the most basic elements for the formulation of all calendar systems on this planet, while lunar calendars are needed to ascertain the festivals and suspicious days of of a year. Solar calendars, on the other hand, account for the periodic revolution of the Earth around the Sun. All these facts were fully understood and recognized during the *Vedāṇga* period.

COUNTING OF TIME

According to *Manusmṛti* (cf. [Dik], 148-149), the duration of Kali Yuga is 4,32,000 years and *Dvāpar*, Treta and Kṛityuga (Satyuga) are two, three and four times of Kaliyuga respectively. These four Yugas form a MahāYuga (Great-Yuga) which covers a time-span of 43, 20,000 years, i.e., 10 times of Kali Yuga. One thousand Mahāyugas form a Kalpa, which is one day of Brahmā. From the beginning of the present Kalpa 453 Mahāyugas have elapsed and out of 454th Mahāyuga the first three Yugas, that is kṛit, Treta and Dvāpar have passed and at present it is Kaliyuga. Majumdar (cf.[Sh], p. 152) has estimated that Kaliyuga started from the year 3102 B.C. Whitney [see Dik, pp. 158-199] too has calculated that Kaliyuga begun from Thursday, the 17^{th} February 3102 B.C., at midnight. According to Sūryasiddhānta too, the present year is the 5103^{rd} year of Kaliyuga, which began from Thursday, Phalguna Krisna 15, at midnight. It may be mentioned here that the Sumerian Great Flood, which destoryed a greater part of the present day Middle East and North-Western Indian plains, occurred in the year 3100 B.C., i.e., about two years after the beginning of Kaliyuga. As a matter

of fact, there is no mention of this great flood in Ṛgveda. But as pointed out by Sharma [Sh, p. 150], Atharvaveda has indicated it in connection with the availability of a medicinal plant in the form:

यत्र नाव प्रमंशनं यत्र हिमवन्तः शिरः।

That is, (the medicinal plant) occurs at the place where the boat glided down on the peak of Himavant (Himalaya).

Since the beginning of Kaliyuga has been mentioned in a number of scriptures, it may be considered as the initial point of the recorded calendar system in India. A calendar in Sanskrit is called " Pañchāng" which means five parts, as it provides information mainly about Tithi (lunar-date), Vasar (day), Nakshatra, Yoga and Karan (Planetary positions). In this section we present a short introduction of some fairly known calendar systems of ancient India.

(i) **Kaliyuga Era :** By adding up 3044 years to Vikram Saṃvat the Kaliyuga era of a time is obtained. At present Kaliyuga year beings from from Caitra Śukla 1 like Vikram Saṃvat

(ii) **Saptarṣikāla :** It is known that Seven sages (सप्तर्षि) pass one constellation in 100 years and complete one round of zodiac in 2700 years. Saptarṣi era was based on the movements of seven sages in the sky. At the time of Alberuni it was used in " Kashmir and Punjab. It used to begin from caitra Śukla 1 like Vikram Saṃvat. At present it is not used anywhere in India.

(iii) **Buddha Nirvaṇa Era :** It begins from Buddha Purnima; i.e., Vaisakha Sukla 15.This year it is 2 5 4 6 Buddha Nirvana era. If we add 487 years to Vikram Saṃvat, we obtain Buddha Saṃvat.

(iv) **Mahāvīra Nirvāṇa Era** : It begins from Kartik Sukla-1. By adding 470 years to Vikram Saṃvat, we get Mahāvīra Nirvāṇa Era.

(v) **Vikram Saṃvat :** It is used in almost every part of India. It begins from caitra Śukla 1, but in Gujarat it begins from Kārtik Śukla 1 (Kartikadi). It

was started by the valiant king, Vikramāditya, of Malwa after defeating the Śaka hordes. By subtracting 57 years and 135 years from Vikram Saṃvat, we obtain Christian and Śaka eras respectively. Due to the great importance of Vikram Saṃvat, we shall discuss its history in detail in a separate section.

(vi) **Christian Era :** Although initiated about 2000 year ago, this era was introduced by British rulers in India as the Official Calendar. It begins from 1st January and its year is solar. The month of February is of 28 days. January, March, May, July, August, October and December months are of 31 days, while April, June, September and November are of 30 days. It was started in Europe 57 years later than Vikram Saṃvat. Now a days it is used throughout India.

(vii) **Śaka Era :** It was initiated 135 years later than Vikram Saṃvat to celebrate the coronation of Kuṣaṇ Emperor Kaniska (cf. [San], p. 203). It is still used in some parts of South India. The Government of India has declared it as the National Calendar. In all other respects it is similar to Vikram Saṃvat.

(viii) **Cedi or Kalcuri Saṃvat :** It was started by the Kalcuri kings of Central India from Asvin Śukla 1 of Vikram Saṃvat 305. It was used in Central and Western part of India during the Kalcuri rule.

(ix) **Gupta Era :** It was initiated to celebrate the coronation of the first Gupta emperor Chandragupta I, from Caitra Sukla 1 of Vikram Saṃvat 376 (319 A.D.). Chandragupta II was the most illustrious emperor of the Gupta dynasty. He defeated Śakas at the battle field of Karur near Multan (cf./ [Sec.], Vol. II, p. 6) and advancing to Vahlika region chased the mighty Huna hordes beyond Vaṃkṣu river, which formed the northern frontier of India. Kalidas has given a vivid description of this battle fought on the south bank of Vaṃkṣhu river in the allegorical form (Cf. [Kal.], Chapter IV, Verses 67 and 68):

विनीताध्व श्रमास्तस्य वंक्षु तीरं विचेष्टनैः
दुधुवुर्वा जिनः स्कन्धान्लग्न कुंकम केसरान्।।
तत्र हूणावरोधानी भर्तृषु व्यक्त विक्रमम्।
कपोल पाटलादेशि वभूवः रघु चेष्टितम्।।

That is, in order to shake off the fatique (of the long march) the horses of Raghu rolled over the field and saffron filaments were seen on their shoulders. There the redness of the cheeks (by beating) of Huna queens, describing the bravery of their deceased husbands, testified the valour of Raghu.

It is said that after firmly securing the North-West frontiers of India against foreign invasions the Emperor, Chandragupta II, performed Yajña (यज्ञ) at the same place in Indraprastha where King Yudhisthira did Rājsuya Yajña for he recognition of his sovereignity.

After this Yajña, Chandragupta II assumed the title of " Vikramāditya " and actively promoted Vikram Saṃvat instead of Gupta era. Vikram Saṃvat, therefore, entirely superceded Gupta era. For other details and references see [Dik], P. 492.

Later on, during the reign of Emperor Budhgupta an Iron Pillar was established at the place of aforesaid Yajna under the supervision of Acharya Varahmihira, the greatest mathematician of that age, to commenorate the great achievements of Chandragupta II " Vikramaditya ". The inscriptions on the pillar is clearly readable of which one line is :

"तीर्त्वा सप्त मुखानि येन समरे सिन्धो र्जिता वाह्लीका।।"

That is, after crossing the river sindh with its seven tributaries, he (Chandragupta) conquered Vahlika (Bactria) region.

(x) **Harṣa Era :** It was initiated by Harṣavarddhana, a renowned king of Kannauj. A number of copper plates marked with this era has ben obtained. By adding 663 years to Harṣa era we obtain Vikram Saṃvat. At the time of

Al Beruni it was being used around Kannauj and Mathura (cf. [Dik], p. 495 for other details).

(xi) **Bengali Saṃvat :** It begins from Vaisakh Krisna 1 and by adding 650 years to it we get Vikram Saṃvat. At present Bengali era is 1409. It is used in Bengal.

(xii) **Kollam Era :** It is used in Kerala. Begins from Sravana Śukla 1 and its year is solarl. By adding 882 years to Kollam era, we obtain Vikram Saṃvat.

(xiii) **Calukya Era :** It was started by Calukya King Vikramaditya from Vikram Saṃvat 1133. It was used in the Calukya kingdom upto Vikram Saṃvat 1219.

(xiv) **Lakshman Sen Era :** It was started y the Sen King, Lakshman Sen, of Eastern Bihar and West Bengal from Vikram Saṃvat 1165. It was used in Central and Eastern Bihar from 13th to 17th Century of Vikram Saṃvat For other details see [Dik], pp. 496-497.

(xv) **Jewish Era :** In order to save from Christian persention jewish people migrated to India during the early Christian era. Jewish settlements are found in some parts of Kerala, Maharastra and Gujarat. They use Jewish Calendar to ascertain their festivals and auspicious days. Jewish year begins towards the end of September and this year it is 5763rd of the Jewish era.

HISTORICITY OF VIKRAM SAṂVAT

In order to count time, a number of calender systems were promulgated in India from time to time, but very few of them could survive beyond a few centruies. Older calendars like Kaliyuga Era (or Yudhisthira Saṃvat), and Maurya Saṃvat were used before the Vikram Saṃvat. Buddha Nirvaṇ and Mahavira Nirvaṇ Saṃvats are used in Buddhist and Jain Scriptures respectively. In the chronological order Vikram Saṃvat comes after Maurya Saṃvat. As mentioned in the preceding section, Vikram Saṃvat was initiated in the year 57 B.C. to

commenorate the great victory of King Vikramaditya against the śaka hordes.

Although some historians have doubted the existence of Vikramaditya and Vikram Saṃvat from 57 B.C., recently a number of research papers have been published to establish the historicity of the Malav K ing, Vikramaditya, and the Saṃvat initiated by him (cf. [Vyas a,b,c], [UPa], [UPb]. We give below the testimony of some impartial Muslim authors about the history of Vikramaditya and Vikram Saṃvat:

(i) **Albernis Evidence :** It is well known in Indian history that about 450 years after king Vikramaditya of Malwa another valiant Emperor, Chandragupta II, of Gupta dynasty also assumed the title of Vikramaditya and defeated Saka hordes at the battle field of Karur near Multan. Alberuni has clealrly distinguished the identity of these two heroes of Indian history (cf. [Sac], Vol. II, P. 6):

" The epoch of the era of śaka or śakala falls 135 years later than that of Vikramaditya. The here-mentioned śaka tyrannised over their country between the river Sindh and the ocean after he had made Aryavarta in the midst of this realm his dwelling place. The Hindus had much to suffer from him, till at last they received heep from the east, when Vikramaditya marched against him, put him to flight and killed him in the region of Karur, between Mustan and the castle of Loni. Now this date became famous, since there is a long interval between the era which is called the era of Vikramaditya and the killing of śaka, we think that the Vikramaditya from whom the era has got its name is not identical with that one who killed śaka, but only a name sake of his."

Abu Rihan Mahammed bin Ahmad al Biruni al Khwarizmi (970-1039 A.D.), who is known in India as 'Alleruni, was a great scholar and traveller. Abul Fazl Baihagi, who lived about half a century after Alberuni, writes in his famous treatise Tawarikhi al-Subuktagini (cf. [ED], p.2]:

" The Master Abu Rihan al Biruni excelled all his contemporarises in the science of philosophy, mathematics and geometry. Sultan of Khwarizṃ appointed

him to accompany the embassy which he sent to Mahmud of Ghazni."

Alberuni, following the army of Mahmud Ghaznvi, came to India and lived here for about ten years. He wrote an extensive account of his study in India in his famous book, Kitab-u-Hindu, which provides a vivid description of the contemporary Indian civilization, culture, science and technology. His testimony that Vikram Saṃvat was initiated by king Vikramāditya 135 years before the saka era provides a clinching argument to settle the problem.

(ii) Evidence in Tabakat-I-Nasiri : The author of Tabakat-I-Nasiri, Abu Umar Minhaj al Jurjani, was a native of Iran. He migrated to India during the reign of Slave doynasty and was appointed as Kazi of Delhi by Sultan Nasiruddin. In Tabakat-I-Nasiri, Jurjani (Juzjani) has given an account of the expedition of Sultan Altamash to conquer Malwa province. He writes (cf. [ED], vol. II, p. 328) :

In 632 H (1234 A.D.) Attamash lead an army against Malwa and took the city and fort of Bhelsa (Vidisha). There was a temple there which was three hundred years in building. It was 105 gaj high. It was demolished from Bhelsa he (Altamash) proceeded to Ujjain where there was the temple of Mahakala which he destroyed as well as an image of Vikramaditya who ruled Malwa 1346 years before this time. The Hindu era dates from his reign (as this figure leads to 1289 A.D. as the date of the capture of Ujjain which really is 1234 A.D. we may take 55 years as the length of Vikrama's rule). Some other images cast in copper were carried to Delhi with the stone image of Mahakala".

The above description of Jurjani implies that Vikramaditya ascended the throne of Malwa in the year (1234-1346) B.C. = 112 B.C. and initiated Vikram Saṃvat in the year 57 B.C. It may be mentioned here that the image of Vikramaditya carried away by Altamash from Mahakala temple was not of Chandragupta II 'Vikramaditya', because the latter was a Vaisnava.

(iii) <u>Evidence in Bābur-Nāmā</u>. According to William Erskine (Cf. [Bab], p. 715):

"Zahīrud-din-Muhammad Bābur was undoubtedly one of the most illustrious men of his age, and one of the most eminent and accomplished princes that ever adorned an Asiatic throne."

Bābur (1483-1530 A.D.) wrote his autobiography in Turkish language. He has given a vivid and extensive description of India including its history, geography, flora and fauna. About the history of Vikramaditya and Vikram Saṃvat Bābur, in 1527, categorically writes (cf. [Bab], p. 79):

"Not more than seven or eight observatories seem to have been constructed in the world. Māmūm Khalīfā made one with which the Māmūnī Tables were written. Batalumūs (Ptolemy) constructed another. Another was made in Hindustan, in the time of Rājā Vikramaditya Hindu, in Ujjain and Dhar, that is, the Malwa country, now known as Mandu. The Hindus of Hindustan use the Tables of this Observatory. They were put together 1584 years ago."

The above quotation from Bābur-Nāmā entails that two Observatories - one at Ujjain and another at Dhar-were constructed in the time of 'Raja Vikramaditya in the year 57 B. C. and a calendar (पंचाङ्ग) was formed on the basis of the data collected from the Observatories of Ujjain and Dhar established by him.

Mahapandit Rahul Sankrityayan, tracing the early history of Śakas in India (cf.[San], pp. 154-155), writes :

"मोग" (Maues) की मृत्यु 58 ई. पू. के बाद भारत में शक राज्य छिन्न-भिन्न हो गया। (यह) विक्रम संवत का आरम्भ समय था।"

It is well known that Mog (or Maues) was a powerful Śakas ruler (77-58 B.C.), who conquered a substantial part of North India including Saurashtra, Ujjayini and Mathura. As pointed out by Rahul Sankrityayan, Mog died in the year 58 B.C. and thereafter the people of Malwa under the command of their valiant leader, Vikramaditya, defeated the Śakas hordes and liberated North India from Śakas tyranny. For other details see [UPa], pp. 16-62.

THE BASIS OF VIKRAM SAMṂVAT

Varahmihira, in Paṇcasiddhāntikā, has given a systematic exposition of all five siddhantas prevalent at his time, namely Paitāmaha, Vasistha, Romaka, Pauliśa and. Sūrya Siddhānta Varahmihira assigns the first place to Sūrya Siddhānta in order of merit. He writes (cf. [Var a]; I,4)

पौलिशकृत: स्फुटोऽसौ तस्य सन्नस्तु रोमक प्रोक्:।
स्पष्टतर: सावित्र: परिशेषौ दूर विभ्रष्टौ।।

That is, the Siddhānta composed by Pauliśa is accurate, Romaka is near to it, more accurate is Sāvitraḥ Sūrya Siddhānta), while the remaining two are far from the truth.

Due to the great importance of Sūrya Siddhānta in astronomical calculations, it has been preserved for more than 2000 years, while the subject matter of other four Siddhāntas are available only in their abridged form presented by Varāhmihīra in his Paṇcasiddhāntikā. However, Sūrya Siddhānta known at the time of Varāḥmihīra was somewhat different from the one availabe at present. Thibaut, for instance in his commentary on Pañcasiddhāntikā writes (cf. [Var a], p XIII):

"A cursory survey of those chapters of the Paṇcasiddhāntikā which treat of the Sūrya Siddhānta shows at once that the treatise of that name known to Varāḥmihīra agrees with the modern Sūrya Siddhānta in its fundamental fetatures. The methods of the two treatises are essentially the same and on the other hand, sufficiently different from those of the other Siddhāntas summarized by Varāḥmihīra, to ensure to the Sūrya Siddhānta in its two fold form a distinct position of its own. At the same time we can not fail to notice that in certain points the teaching of the old Sūrya Siddhānta (by which name I shall, for shortness sake, designate the Sūrya Siddhānta known to Varāḥmihīra, must have differed from the corresponding doctrines of its modern representative.

Sūrya Siddhānta was composed completely on the basis of traditional Indian astronomy, being different in certain features from Paulisa and Romaka

Siddhāntas. Brahmagupta, in fact, has blamed Romaka Siddhānta for not following the traditional Hindu methods in some respects (for details see [Var a], p. LX VIII). Although some authors point out the influence of greek astronomy, Thibaut, on the other hand, emphasizes : [loc. at., pp. LXXIV-LXXV]:

"A work of the class of the Sūrya Siddhānta should spring directly from a work such as the Syntaxis, would be almost incomprehensible........The merit of (its) originality, as for as it goes, would most probably belong to the unknown author of the old Sūrya Siddhānta."

From the above quotation it is clear that Sūrya Siddhānta presents an outgrowth of traditional Indian Siddhāntas without any direct link with the Greek astronomy. Although Thibaut attributes the composition of Siddhāntas to an "unknown author", in the beginning of modern Sūrya Siddhānta a reliable introduction of its original author is given in the form (cf. [SS], I, Coplets 2-4):

अल्पावशिष्ट तुकृते मयनामा महा सुरः।
रहस्यं परमं पुण्य जिज्ञासुर्ज्ञान मुत्तमम्।।2।।
वेदांगम्ग्रयमऽखिलं ज्योतिषांगतिकारणम्।
आराध्ययन्वित स्वन्तं तपस्ते पेसुदुश्चरम्।।3।।
तोषितस्तपसातेन प्रतिस्तस्मै वरार्थिमने।
ग्रहाणां चरितं प्रादान्मयाय सविता स्वयम्।।4।।

The above conplets state that when a small (part) of Kṛit(Saṃvat) remained a great Asur, named Maya, inquisitive of the meritorious (holy) secrets of Jyotiṣa (astronomy), which is the foremost part of Vedāṅga, studied the Sun with inaccessible ascetic fervour. The Sun (God) satisfied with his boon-aspiring asceticism provided him the characters of the planets.

According to Thibaut (cf. [Vara], p.L.XXXIII), the Romaka Siddhānta was not composed later than 400 A.D. Dikshit, on the other hand, using some astronomical calculations (cf. [Dik], p 221) has inferred that the original Romaka Siddhānta was composed after Hipparchus (150 B.C.), but before the time of Ptolemy (150 A.D.). He further asserts (loc. cit., p.222) that Sūryasiddhānta was

composed earlier than Romaka. (cf.[Vyd], pp. 170-172).

He has mentioned that during an excavation of old mounds at Ujjain in the year 1938-1939 an old coin marked Maya (मय) was obtained.

Vyas writes that Mayasur was a resident of Ujjain, who composed Sūryasiddhānta to rectify a number of defects in the traditional Indian Siddhantas in the first century before the Christian era. Prior to 57 B.C. an old calendar, named Kṛit Saṃvat, was used in Malwa and a number of inscriptions and coins marked, Krit or / and Malav have been obtained. Vyas maintains that at the behest of King Vikramaditya, Mayasur thoroughly revised the old Kṛit Saṃvat and formulated a new one called 'Vikram Saṃvat, Although Kṛit Saṃvat was in use for a few centuries more, Vikram Saṃvat took its place in due course. C.V. Vaidya (cf.[Vy d], p.39) too supports this assertion and claims that Vikram Saṃvat has been formulated on the basis of Sūryasiddhānta. It may be mentioned have that Varāhmihīra has included the name of Maya among the astronomers of ancient India (cf. [vy d], p. 172 in the form :'

मय - यवत - मणित्थ शक्तिपूर्वे
र्दिवस करादिषु वासराः प्रदिष्टाः"

EPILOGUE

The science of astronomy was cultivated in India from the time of Ṛgveda. The Vedic scholars discovered that the Sun light was composed of seven rays, and one year consisted of 12 month and each month was of 30 days. A total list of 27 Nakshatras (constellations) are given in Taittiriyā Saṃhitā. The concept of a Yuga consisting of five Saṃvatsaras (years) was recognized during the Vedāṅga period. The name of these Saṃvatsaras are given in Taittiriyā Brahmaṇ Vedāṅga Jyotisa, which was composed by Lagadh around 1400 B.C., provides a more advanced aspect of the ancient Indian calendar. Yajur Jyotiṣa for instance, asserts that a Saṃvatsara consists of 12 solar months, six seasons and 366 days, and a Yuga is five times of a Saṃatsara. Vedanga Jyotiṣa has clearly made distinction between the Savana solar and lunar months.

Manusmṛti on the other hand, has given the concept of a Mahāyuga consisting of Krita, Treta, Dvāpar and Kaliyuga. According to Sūryasiddhānta the present year is the 510 3rd year of Kaliyuga which had began from Thursday, Phālguna Krisṇa 15, at midnight, that is, Kaliyuga had started from 17th February 3102 B.C. at midnight.

Although 15 calendar systems were introduced in ancient India from time to time, only Vikram Saṃvata and Christian era have survived. Vikram Saṃvata, which is based on the formulation of Sūryasiddhānta is being used with some variations in almost every part of India. It is really a great anomaly and totally improper that the Government of India has recognized alien Śakas Saṃvata as the National Calendar ignoring the claim of indigenous Vikrama Saṃvata

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STOCHASTIC MODEL OF A POPULATION GROWTH WITH PERIODICALLY VARYING PARAMETERS

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ABSTRACT

The paper deals with the study of the stochastic behaviour of a population obeying Gompertizian growth equation with both periodic growth coefficient and periodic carrying capacity under the influence of randomly fluctuating environment.

Mathematics subject classification (2000) : 92A17

Key words: Gompertz equation, Harmonic growth coefficient, White noise, Stochastic process, Periodic carrying capacity, Transfer function.

INTRODUCTION

The growth plays a significant role in life-process. There are different models of growth processes. The Malthusian [4] growth for which the growth coefficient α constant is meaningless after a large time t due to the ridiculously large value of the population size. An important generalisation of Malthusian model is the Gompertz model equation which finds wide application in different growth processes of men, animals, plants, tumour etc. [3] . Gompertz model equation is given by [1,2,3].

$$\frac{dN(t)}{dt} = -\alpha N(t) \ln \frac{N(t)}{K} \qquad \dots \quad \dots \qquad (1.1)$$

with $N(0) = N_0, N(\infty) = K$.

Writing $x(t) = lnN(t)$, the equation (1.1) reduces to

$$\frac{dx(t)}{dt} = -\alpha x(t) + \alpha \ln K \qquad \dots \quad \dots \qquad (1.2)$$

where $$N(t) \geq 1 \Rightarrow x(t) \geq 0 \qquad \dots \quad \dots \qquad (1.3)$$

In the above equation (1.1) or (1.2), α is the growth coefficient and K is the carrying capacity [1,2]. A great many phenomena are cyclic or oscillatory in behaviour. Due to diurnal or annual influences, there are numerous environmental factors like temperature, sunlight etc. which affect the growth processes. As a result cyclic variations occur in the growth process and the growth coefficient α and carrying capacity may vary periodically with time. Besides, the randomly fluctuating environment effects on the growth processes. The object of the present paper is to study the overall effects of the periodically varying growth coefficient or carrying capacity and the randomly fluctuating environment on the growth behaviours of a population obeying Gompertzian model equation.

GOMPERTZIAN GROWTH IN RANDOM ENVIRONMENT : SINUSOIDALLY VARYING GROWTH COEFFICIENMT

We shall start with a simple harmonic growth coefficient and take [1].

$$\alpha(t) = a_0 + a\sin(\omega t + \varphi), a_0, a < 0 \qquad \ldots \quad \ldots \quad (2.1)$$

This says that the growth coeficient, $\alpha(t)$, has an average value, a_0, and oscillates with amplitude, a, about this average value at a frequency, $\omega = 2\pi / T$; T is the period of oscillation and φ is the phase angle. The oscillation increases as a increases and no oscillation for $a = 0$. The phase angle φ is : $0, \pi/2, \pi, 3\pi/2$. For small oscillation in $\alpha(t)$ a and a_0 satisfy the condition:

$$|a / a_0| << 1. \qquad \ldots \quad \ldots \quad (2.2)$$

Putting equation (2.1) in equation (1.2) we get

$$\frac{dx(t)}{dt} + \alpha(t)x(t) = \alpha(t) \ln K = F_1(t) \quad \text{(say)} \qquad \ldots \quad \ldots \quad (2.3)$$

where $F_1(t)$ is given by

$$F_1(t) = \{a_0 + a \sin(\omega t + \varphi)\} \ln K \qquad \ldots \quad \ldots \quad (2.4)$$

To take account of the influence of the randomly fluctuating environment we add in equation (2.3) a white noise (i.e. fluctuating term) $f(t)$ whose properties are given by [8]

$$\langle f(t) \rangle = 0, \langle f(t) f(t') \rangle = 2D(t)\delta(t - t') \qquad \ldots \quad \ldots \quad (2.5)$$

where $\langle \ \rangle$ denotes the average over the ensemble of the stochastic process, $D(t)$ is the intensity (or diffusion coefficient) of the white noise. Then the equation (2.3) becomes

$$\frac{dx(t)}{dt} + \alpha(t)x(t) = F_1(t) + f(t) = \beta_1(t). \quad \text{(say)} \qquad \ldots \quad \ldots \quad (2.6)$$

where

$$\langle \beta_1(t) \rangle = \langle F_1(t) + f(t) \rangle$$

$$= F_1(t) \text{ and } \langle \beta_1(t)\beta_1(t') \rangle = F_1^{\,2}(t) + 2D(t)\delta(t - t') \qquad \ldots \quad \ldots \quad (2.7)$$

The solution of the equation (2.6) can be written as [6]

$$x(t) = \int_0^t g_1(t,\tau)\beta_1(\tau) d\tau = \int_0^t g_1(t,\tau)[F_1(t) + f(\tau)] d\tau \qquad \ldots \quad \ldots \quad (2.8)$$

where the transfer function $g_1(t,\tau)$ is given by [6]

$$g_1(t,\tau) = 0, \tau < t, \quad g_1(\tau,\tau) = 1 \text{ and}$$

$$\frac{dg_1(t,\tau)}{dt} = -\alpha(t) \quad g_1(t,\tau) \qquad \dots \quad \dots \qquad (2.9)$$

The equation (2.9) leads to

$$g_1(t,\tau) = e^{-\int_\tau^t \alpha(t)dt} = e^{-a_0 t + \frac{a}{\omega}cos(\omega t+\varphi) + a\tau - \frac{a}{\omega}cos(\omega\tau+\varphi)} \qquad \dots \quad \dots \qquad (2.10)$$

The variance of the stochastic process $x(t)$ is

$$\sigma_1^2(t) = < x^2(t) > - < x(t) >^2$$

$$= \int_0^t\int_0^t g_1(t,\tau)g_1(t,\tau') < \beta_1(\tau)\beta_1(\tau') > d\tau d\tau'$$

$$= \int_0^t g_1^2(t,\tau)\left[F_1^2(\tau) + 2D(\tau)\right]d\tau \qquad \dots \quad \dots \qquad (2.11)$$

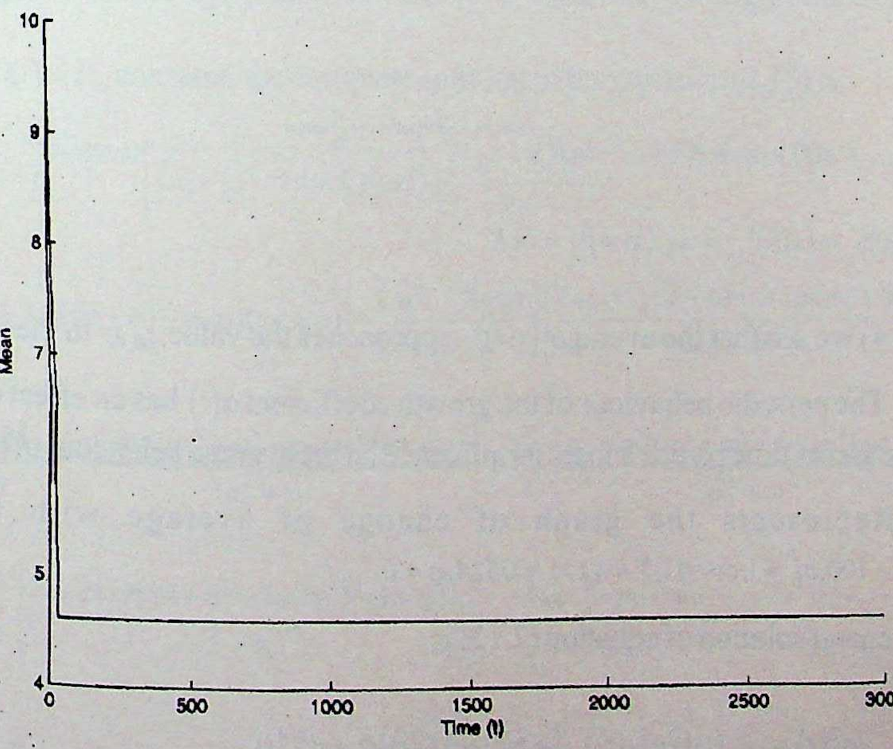

Fig. 1: Changes of averages $\langle x(t) \rangle_1$ with time t in days for sinusoidally varying growth coefficient.

Differentiating equation (2.11) w.r. to time t[7] we have

$$\frac{d\sigma_1^2(t)}{dt}+2\alpha(t)\sigma_1^2(t)=F_1^2(t)+2D(t) \quad \dots \quad \dots \quad (2.12)$$

which is the basic equation of the time evolution of the variance of the stochastic process $x(t)$ and $F_1(t)$ is given by equation (2.4). The average value of $x(t)$ is $< x(t) >$, then equation (2.6) gives

$$\frac{d< x(t) >_1}{dt}+\alpha(t)< x(t) >_1=F_1(t) \quad \dots \quad \dots \quad (2.13)$$

with initial condition $< x(0) >_1=x_0$ (say),

The complete solution of equation (2.13) is

$$< x(t) >_1=< x(t) >_{1h}\int_0^t \left\{< x(t) >_{1h}\right\}^{-1} F_1(t)+< x(t) >_{1h}$$

or,

$$< x(t) >_1=ln\,K+(x_0-ln\,K)e^{-a_0t+\frac{a}{\omega}cos(\omega t+\varphi)-\frac{a}{\omega}cos\varphi} \quad \dots \quad \dots \quad (2.14)$$

where $\langle x(0)\rangle_1=\langle x(0)\rangle_{1h}=x_0, \langle x(\infty)\rangle_1=ln\,K$

From (2.14) we see that the average $\langle x(t)\rangle_1$ approaches the value $ln\,K$ in the long run as expected. The periodic behaviour of the growth coefficient $\alpha(t)$ has an effect only in the initial state and as time passes it loses its influence on the average behaviour of the system. Fig. I Represents the graph of change of average with time for $x_0=10, K=100, a_0=1, a=0.1, T=12, \omega=0.524, \varphi=0$.

General solution of equation (2.12) is

$$\sigma_1^2(t)=\sigma_{1h}^2(t)\int_0^t\left\{\sigma_{1h}^2(t)\right\}^{-1}\left[F_1^2(t)+2D(t)\right]dt+\sigma_{1h}^2(t) \quad \dots \quad \dots \quad (2.15)$$

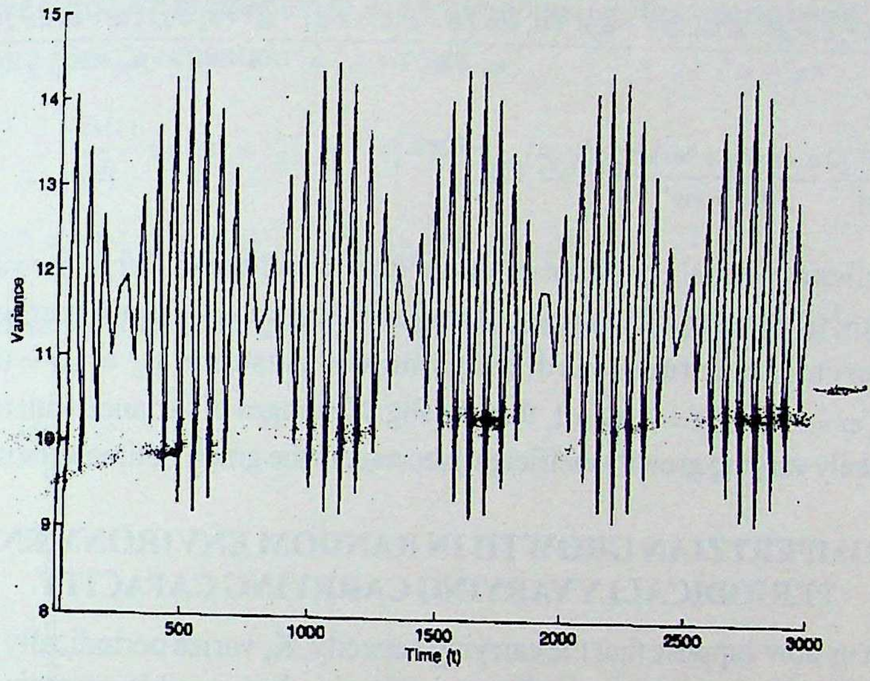

Fig. 2: Changes of variance $\sigma_2^2(t)$ with time t in days for sinusoidally varying growth coefficient.

Assuming $D(t) = D$, constant, the complete solution of the equation (2.15) is

$$\sigma_1^2(t) = e^{\frac{2a\cos(\omega t+\varphi)}{\omega}} \left[\left\{ a_0^2 + (a^2/2)(\ln K)^2 2D \right\} \frac{1}{2a_0} + \frac{2a_0 a}{4a_0^2+\omega^2} \times \right.$$

$$\times \left\{ 2a_0 \sin(\omega t+\varphi) - \omega \cos(\omega t+\varphi) \right\} - \left(\frac{a^3}{2\omega} + \frac{2a_0^2 a}{\omega} \right) \left\{ \frac{2a_0 \cos(\omega t+\varphi) + \omega \sin(\omega t+\varphi)}{4a_0^2+\omega^2} \right\} -$$

$$-\frac{a_0 a^2}{\omega} \times \left\{ \frac{a_0 \sin 2(\omega t+\varphi) - \omega \cos 2(\omega t+\varphi)}{a_0^2+\omega^2} \right\} - \frac{a^2}{4} \left\{ \frac{a_0 \cos 2(\omega t+\varphi) + \omega \sin 2(\omega t+\varphi)}{a_0^2+\omega^2} \right\} +$$

$$\left. + \frac{a^3}{2\omega} \left\{ \frac{2a_0 \cos 3(\omega t+\varphi) + 3\omega \sin 3(\omega t+\varphi)}{4a_0^2+9\omega^2} \right\} \right] - e^{-2a_0 t + \frac{2a}{\omega}\cos(\omega t+\varphi)} \times$$

$$\times \left[\left\{ \left(a_0^2 + a^2/2\right)(\ln K)^2 + 2D \right\} \frac{1}{2a_0} + \frac{2a_0 a(2a_0 \sin\varphi - \omega\cos\varphi)}{4a_0^2+\omega^2} - \left(\frac{a^3}{2\omega} + \frac{2a_0^2 a}{\omega} \right) \times \right.$$

$$\times\left(\frac{2a_0\cos\varphi+\omega\sin\varphi}{4a_0^2+\omega^2}\right)-\frac{a_0a^2(a_0\sin 2\varphi-\omega\cos 2\varphi)}{a_0^2+\omega^2}-\frac{a^2}{4}\frac{(a_0\cos 2\varphi+\omega\sin 2\varphi)}{a_0^2+\omega^2}+$$

$$\left.+\frac{a^3}{2\omega}\frac{(2a_0\cos 3\varphi+3\omega\sin 3\varphi)}{4a_0^2+9\omega^2}-\sigma^2(0)e^{-\frac{2a\cos\varphi}{\omega}}\right] \quad \ldots \quad \ldots \quad (2.16)$$

which satisfies the initial condition $\sigma_1^2(0)=\sigma^2(0)$. When time t is large the variance (or fluctuation) $\sigma_1^2(t)$ remains finite and oscillatory with superpositions of a number of waves with different amplitudes and frequencies. Assume $a_0=1, a=0.1, T=12$, $\omega=0.524, \varphi=0, \sigma^2(0)=2, D=1$, then the Fig.2 (changes of variance with time in days for sinusoidally varying growth coefficient) represents the graph of time versus variance.

GOMPERTZIAN GROWTH IN RANDOM ENVIRONMENT: PERIODICALLY VARYING CARRYING CAPACITY

Let us now suppose that the carrying capacity, K, varies periodically with time t because of either diurnal or annual influences. The simplest possible example of carrying capacity oscillatory with angular frequency ω is obtained by assuming [5].

$$K(t)=K_0(1+b\cos\omega t) \quad \ldots \quad \ldots \quad (3.1)$$

where K_0 is the average value of K over a long time interval and provided

$$0<b<<1 \quad \ldots \quad \ldots \quad (3.2)$$

i.e., provided the fluctuations in K are small relative to K_0. The equation (3.1) can be written as:

$$K(t)=K_0e^{b\cos\omega t} \quad \ldots \quad \ldots \quad (3.3)$$

provided the condition (3.2) is satisifed.

Due to periodic effect in carrying capacity $K(t)$ given by equation (3.1), the growing population size varies. Putting equation (3.1) in equation (1.2) taking growth coefficient α constant we get:

$$\frac{dx(t)}{dt}+\alpha x(t)=\alpha\, ln\, K(t)=F_2(t), \quad \text{(say)} \quad \ldots \quad \ldots \quad (3.4)$$

where $$F_2(t)=\alpha(ln\, K_0+b\cos\omega t) \quad \ldots \quad \ldots \quad (3.5)$$

As before we introduce in equation (3.4) a white noise (i.e., fluctuating term) $f(t)$ whose properties are given by equation (2.5), we get:

$$\frac{dx(t)}{dt} + \alpha x(t) = F_2(t) + f(t) = \beta_2(t), \quad \text{(say)} \qquad \dots \quad \dots \qquad (3.6)$$

where $\langle \beta_2(t) \rangle = F_2(t), \langle \beta_2(t)\beta_2(t') \rangle = F_2^2(t) + 2D(t)\delta(t-t')$ ... ... (3.7)

We can take suitable function $g_2(t,\tau)$, whose properties are same as $g_1(t,\tau)$, in section (2). Then

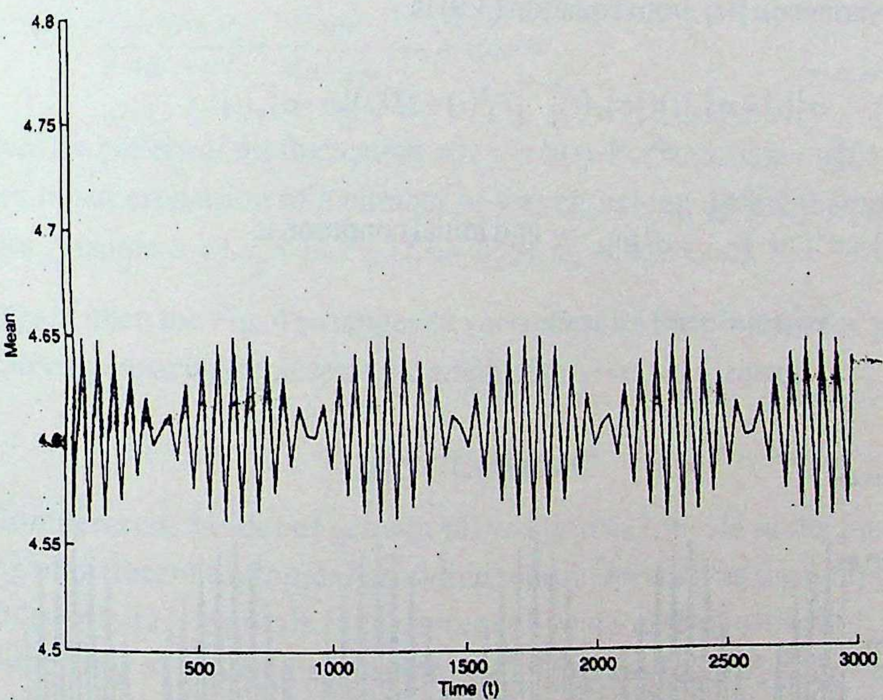

Fig. 3: Changes of average $\langle x(t) \rangle_2$ with time t in days for periodically varying carrying capacity.

$$g_2(t,\tau) = e^{-\int_\tau^t \alpha dt} = e^{-\alpha t + \alpha\tau} \qquad \dots \quad \dots \qquad (3.8)$$

According to the section (2) we get the variance equation for the stochastic process $x(t)$ where $F_2(t)$ is given by equation (3.5). The average value $\langle x(t) \rangle_2$ of $x(t)$ is given:

$$\frac{d < x(t) >_2}{dt} + \alpha < x(t) >_2 = F_2(t), < x(0) >_2 = x_0, \quad \text{(say)} \qquad \dots \quad \dots \qquad (3.9)$$

The general solution equation (3.9) is

$$< x(t) >_2 = ln\, K_0 + \frac{\alpha b(\alpha \cos \omega t + \omega \sin \omega t)}{\alpha^2 + \omega^2} + \left(x_0 - ln\, K_0 - \frac{\alpha^2 b}{\alpha^2 + \omega^2} \right) e^{-\alpha t} \quad \ldots \quad (3.10)$$

From (3.10) we see that the average $\langle x(t) \rangle_2$ remains finite always i.e., the poplation size $N(t)$ remains finite always and after a large time, the population shows periodic behaviour.

Fig.3. Represents the graph of time verses average $\langle x(t) \rangle_2$ for $x_0 = 10, \alpha = 1, \alpha = 1, T = 12, \omega = 0.524, K_0 = 100, b = 0.05.$ Now according to section (2) the solution for variance $\sigma_2^2(t)$ from equation (3.9) is

$$\sigma_2^2(t) = \sigma_{2h}^2(t) \int_0^t \left\{ \sigma_{2h}^2(t) \right\}^{-1} \left[F_2^2(t) + 2D(t) \right] dt + \sigma_{2h}^2(t) \quad \ldots \quad \ldots \quad (3.11)$$

where $\sigma_{2h}^2(t) = \sigma^2(0) e^{-\int_0^t 2\alpha dt} = \sigma^2(0) e^{-2\alpha t}$ and initial condition is

$$\sigma_{2h}^2(0) = \sigma^2(0), \quad \text{(say)} \quad \ldots \quad \ldots \quad (3.12)$$

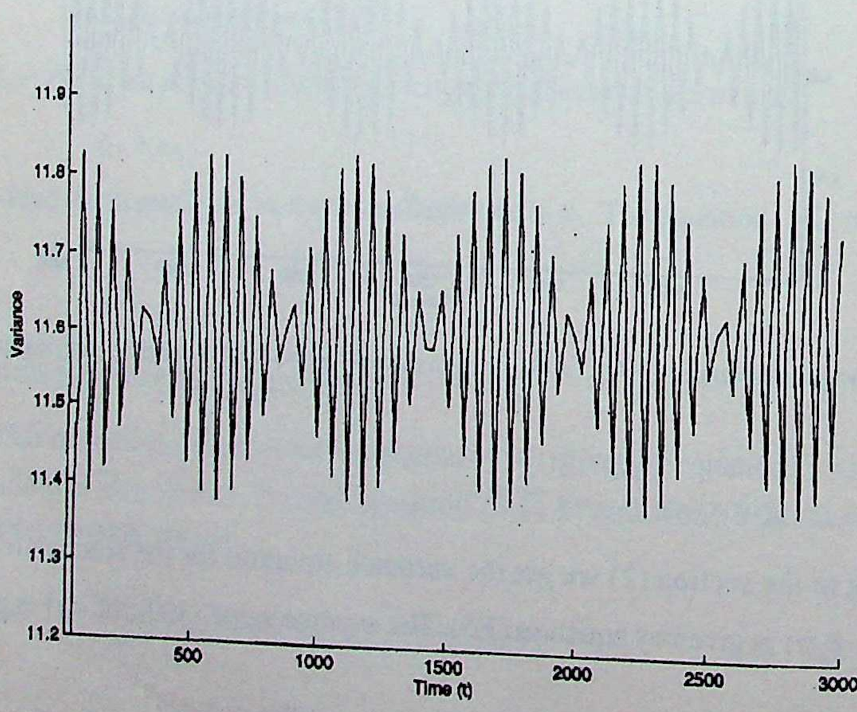

Fig. 4: Changes of variance $\sigma_2^2(t)$ with time t in days for periodically varying carrying capacity.

Assuming $D(t) = D$ constant, then equation (3.12) gives

$$\sigma_2^2(t) = e^{-2\alpha t}\left[\alpha^2\int_0^t (ln\, K_0 + b\cos\omega t)^2 e^{2\alpha t}dt + 2D\int_0^t e^{-2\alpha t}dt\right] + \sigma^2(0)e^{-2\alpha t} \quad \ldots \quad (3.13)$$

or, $$\sigma_2^2(t) = \frac{\alpha}{2}(ln\, K_0)^2 + \frac{b^2}{4\alpha} + \frac{D}{\alpha} + \frac{2\alpha^2 b\, ln\, K_0}{4\alpha^2 + \omega^2}(2\alpha\cos\omega t + \omega\sin\omega t) +$$

$$+\frac{b^2(\alpha\cos 2\omega t + \omega\sin 2\omega t)}{4(\alpha^2+\omega^2)} - \left\{\frac{\alpha}{2}(\ln K_0)^2 + \frac{b^2}{4\alpha} + \frac{D}{4\alpha} +\right.$$

$$\left. + \frac{4\alpha^3 b\ln K_0}{4\alpha^2+\omega^2} + \frac{\alpha b^2}{4(\alpha^2+\omega^2)} - \sigma^2(0)e^{-2\alpha t} \quad \ldots \quad \ldots \quad (3.14)\right.$$

Which gives the pattern of the fluctuation $x(t) = ln\, N(t)$. For large time $t, \sigma_2^2(t)$ is finite and oscillatory of superposition of a number of waves, having different amplitudes and frequencies. Assume $\alpha = 1, x_0 = 10, T = 12, \omega = 0.524, K_0 = 100, b = 0.05,$

$D = 1, \sigma^2(0) = 2,$ then the Fig. 4 (changes of variance with time in days for periodically varying carrying capacity) represents the graph of time versus variance.

CONCLUSION

Gompertzian Model of growth plays significant role in the mathematical modelling of different biological growth phenomena such as turmour growth [3], growth of plants [2], animals [12]. In recent years non-equilibrium thermodynamic modelling of biological growth is in progress [13,14]. Non-equilibrium thermodynamic theory of biological growth on the basis of Gompertz model equation was developed by Chakrabarti and Bhadra [11], Sarkar and Bhaumik [10].

In this paper we have investigated the characteristic stochastic properties of the Gompertzian growth model for two cases. First one is cyclically growth coefficient having constant carrying capacity with time. Second one is periodically varying carrying capacity having constant growth coefficient with time. In both cases changes of averages and variances (fluctuations) with time have been analysed and the patterns have been shown graphically. Since *x(t)* is logarithmic and monotonically increasing function of the population size the conclusion drawn for *x(t)* also holds good for the population size *N(t)*.

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THREE FIXED POINT REULTS FOR THREE MAPS

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ABSTRACT

We obtain a fixed point Theorem in three different spaces and two more fixed point Theorems in a space using three different non-negative functions.

Key Words : Fixed point, complete metric space, commutativity, non-negative Function, contractive conditions.

2000 Mathematics subject classification : 47H10, 54H25.

INTRODUCTION

We obtain three fixed point theorems using three non-negative functions for three maps satisfying Nung [1], Jain et al. [2] type Contractive conditions.

Nung [1] proved

THEOREM 1 :

Let (X,d), $(X,d),(Y,\rho)$ and (Z,σ) be complete metric spaces and $T: X \to Y, S: Y \to Z, R: Z \to X$ be continuous maps satisfying

$$d\ (RSTx, RSY) \leq c\ max\ \{d(x, RSy), d(x, RSTx), \rho(y, Tx), \sigma(Sy, STx)\} \tag{1.1}$$

$$\rho(TRSy, TRz), \geq\ c\ max\ \{\rho(y, TRz), \rho(y, TRSy), \sigma(z, Sy)), d(Rz, RSy)\} \tag{1.2}$$

$$\sigma(STRz, STx) \leq\ c\ max\ \{\sigma(z, STx), \sigma(z, STRz), d(x, Rz), \rho(Tx, TRz)\} \tag{1.3}$$

for all $x \in X, y \in Y$ and $z \in Z$ where $0 \leq c < 1$. Then *RST* has a unique fixed point *u* in *X*, *TRS* has a unique fixed point *v* in *Y* and *STR* has a unique fixed point *w* in *Z*. Further $Tu = v$, $Sv = w$ and $Rw = u$.

Jain, et al. [2] proved

THEOREM 2 :

Let(X,d),(Y,ρ) and (Z,σ) be complete metric spaces and $T: X \to Y, S: Y \to Z, R: Z \to X$ be maps satisfying

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$$d^2(RSy, RSTx) \leq c \max \{d(x, RSy)\rho(y, Tx), \rho(y, Tx)d(x, RSTx),$$

$$d(x, RSTx)\sigma(Sy, STx), \sigma(Sy, STx)d(x, RSY)\} \quad (2.1)$$

$$\rho^2(TRz, TRSy) \leq c \max \{\rho(y, TRz)\sigma(z, Sy), \sigma(z, Sy)\rho(y, TRSy)$$

$$\rho(y, TRSy)d(Rz, RSy), d(Rz, RSy)\rho(y, TRz)\} \quad (2.2)$$

$$\sigma^2(STx, STRz) \leq c \max \{\sigma(z, STx)d(x, Rz), d(x, Rz)\sigma(z, STRz),$$

$$\sigma(z, STRz)\rho(Tx, TRz), \rho(Tx, TRz)\sigma(z, STx)\} \quad (2.3)$$

for all $x \in X, y \in Y$ and $z \in Z$ where $0 \leq c < 1$. If one of R, S, T is continuous then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further $Tu = v$, $Sv = w$ and $Rw = u$.

MAIN RESULTS

First we prove a fixed point Theorem in three different spaces.

THEOREM 3 :

Let X, Y, Z be three non-empty spaces, 'd', 'ρ', 'σ' be three non-negative functions from $X \times X$, $Y \times Y$ and $Z \times Z$ such that $d(x_1, x_2) = 0$ iff $x_1 = x_2, \rho(y_1 y_2) = 0$ iff $y_1 = y_2$ and $\sigma(z_1, z_2) = 0$ iff $z_1 = z_2$. Further assume that $T: X \to Y, S: Y \to Z, R: Z \to X$ be satisfying

$$d(RSy, RSTx) < \max \{d(x, RSy), d(x, RSTx), \rho(y, Tx)\} \quad (3.1)$$

$$\rho(TRz, TRSy) < \max \{\rho(y, TRz), \rho(y, TRSy), \sigma(z, Sy)\} \quad (3.2)$$

$$\sigma(STx, STRz) < \max \{\sigma(z, STx), \sigma(z, STRz), d(x, Rz)\} \quad (3.3)$$

for all $x \in X, y \in Y, z \in Z$ with $y \neq Sy, x \neq Rz$ respectively.

If the function $x \to \rho(y, TRSy)$ attains its infimum on Y or the function

$z \to \sigma(z, STRz)$ attains its infimum on Z then RST or TRS or STR has a unique fixed point.

PROOF :

Suppose the function $x \to d(RSTx, x)$ attains its infimum on X.

That is there exists a $u \in X$ such that $d\,(u,RSTu) = \text{Inf}\,\{d(x, RSTx) / x \in X\}$

$$= f(u)(say)$$

Suppose RST, TRS and STR have no fixed points.

From (3.1) we have

$f(RSTRST\,u) = d\,(RSTRSTu,\ RSTRSTRST\,u)$

$$< max\{d(RSTRSTu, RSTRSTu), d(RSTRSTu, RSTRSTRSTu), \rho(TRSTu, TRSTRSTu)$$

$$= \rho(TRSTu, TRSTRSTu)$$

From(3.2) we have

$$\rho(TRSTu, TRSTRSTu) < max\ \{p(TRSTu, TRSTu), \rho(TRSTu, TRSTRSTu),$$
$$\sigma(STU, STRSTU)\}$$

From (3.3) We have

$$\sigma(STu, STRSTu) < max\{\sigma(STu, STu), \sigma(STu, STRSTu), d(u, RSTu)\}$$
$$= d(u, RSTu) = f(u)$$

Thus $f\,(RSTRSTu) < f\,(u)$, which is a contradiction.

Hence either RST or TRS or STR has a fixed point.

Similarly we can prove either RST or TRS or STR has a fixed point when either $y \to \rho(y, TRSy)$ or $z \to \sigma(z, STRz)$ attains infimum.

Suppose RST has two fixed points say u, u'.

$$d(u,u') = d(RSTu, RSTu') < \max\{d(u',u), d(u'u'), \rho(Tu, Tu')\}$$

$$= \rho(Tu, Tu')$$

$$\rho(Tu, Tu') = \rho(TRSTu, TRSTu')$$

$$< \max\ \{\rho(Tu',Tu), \rho(Tu',Tu'), \sigma(STu, STu')\}$$

$$= \sigma(STu, STu')$$

$$\sigma(STu, STu') = d(STRSTu, STRSTu')$$

$$< \max\ \{\sigma(STu', STu), \sigma(STu', STu'), d(u,u')\}$$

$$= d(u,u')$$

Then $d(u,u') < d(u,u')$, a contradiction.

Thus RST has a unique fixed point.

REMARK 4 :

Theorem 3 holds if (3.1), (3.2), (3.3) are replaced by

$$d^2(RSy, RSTx) < d(x, RSy)d(x, RSTx) + d(x, RSTx)\rho(y, Tx)$$
$$+\mathrm{d}\ (x, RSY)\sigma(Sy, STx) + d(x, RSy)\rho(y, Tx) \tag{4.1}$$

$$\rho^2(TRz, TRSy) < \rho(y, TRz)\rho(y, TRSy) + \rho(y, TRSy)\sigma(z, Sy)$$
$$+\rho(y, TRz)d(Rz, RSy) + \rho(y, TRz)\sigma(z, Sy) \tag{4.2}$$

$$\sigma^2(STx, STRz) < \sigma(z, STx)\sigma(z, STRz) + \sigma(z, STRz)d(x, Rz)$$
$$+\sigma(z, STx)\rho(Tx, TRz) + \sigma(z, STx)d(x, Rz) \tag{4.3}$$

REMARK 5 :

Theorem 3 holds if (3.1), (3.2) and (3.3) are replaced by

$$d^3(RSy, RSTx) < max\{d(x, RSy)d^2(x, RSTx) + d^2(x, RSy)d(x, RSTx)$$

$$+d(x. RSy)\rho^2(y, Tx) + d^2(x, RSy)\rho(y, Tx)$$

$$+d(x,. RSy)\sigma^2(Sy, STx) + d^2(x, RSy)\sigma(Sy, STx)$$

$$+d(x, RSy)d(x, RSTx)\rho(y, Tx)$$

$$+d(x, RSy)\rho(y, Tx)\sigma(Sy, STx)$$

$$+d(x, RSy)d(x, RSTx)\sigma(Sy, STx),$$

$$d^2(x, RSTx)\rho(y, Tx), d(x, RSTx)\rho^2(y, Tx) \tag{5.1}$$

$$\rho^3(TRz, TRSy) < Max\{\rho(y, TRz)\rho^2(y, TRSy) + \rho^2(y, TRz)\rho(y, TRSy)$$

$$+\rho(y, TRz)\sigma^2(z, Sy) + \rho^2(y, TRz)\sigma(z, Sy)$$

$$+\rho(y, TRz)d^2(Rz, RSy) + \rho^2(y, TRz)d(Rz, RSy)$$

$$+\rho(y, TRz)\rho(y, TRSy)\sigma(z, Sy)$$

$$+\rho(y, TRz)d(Rz, RSy)\sigma(z, Sy)$$

$$+\rho(y, TRz)\rho(y, TRSy)d(Rz, RSy),$$

$$\rho^2(y, TRSy)\sigma(z, Sy), \rho(y, TRSy)\sigma^2(z, Sy)\} \tag{5.2}$$

$$\sigma^3(STx, STRz) < Max\{\sigma(z, STx)\sigma^2(z, STRz) + \sigma^2(z, STx)\sigma(z, STRz)$$

$$+\sigma(z, STx)d^2(x, Rz) + \sigma^2(z, STx)d(x, Rz)$$

$$+\sigma(z, STx)\rho^2(Tx, TRz) + \sigma^2(z, STx)\rho(Tx, TRz)$$

$$+\sigma(z,STx)\sigma(z,STRz)d(x,Rz)$$

$$+\sigma(z,STx)d(x,Rz)\rho(Tx,TRz)$$

$$+\sigma(z,STx)\sigma(z,STRz)\rho(Tx,TRz),$$

$$\sigma^2(z,STRz)d(x,Rz),\sigma(z,STRz)d^2(z,Rz)\} \quad (5.3)$$

Now we give a Theorem on a set equipped with three different non-negative functions.

THEOREM 6 :

Let X be a non-empty set, $S,T,R: X \to X$ and $d, \rho\ \sigma$ be three non-negative functions from $X \times X$ such that $d(x_1 x_2)=0$ iff $x_1 = x_2$, $\rho(x_1 x_2)=0$ iff $x_1 = x_2$ and $\sigma(x_1, x_2)=0$ iff $x_1 = x_2$. Further assume that

$$d(STx,TRy) < \max\ \{\rho(Tx,Ry), d(STx,Tx), \rho(TRy,Ry), \sigma(Tx,TRy)\} \quad (6.1)$$

$$\rho(TRx,RSy) < \max\ \{\sigma(Rx,Sy), \rho(TRx,Rx), \sigma(RSy,Sy), d(Rx,RSy)\} \quad (6.2)$$

$$\sigma(RSy,STx) < \max\ \{d(Sy,Tx), \sigma(RSy,Sy), d(STx,Tx), \rho(Sy,STx)\} \quad (6.3)$$

for all $x, y \in X$ with $Tx \neq Ry$, $Rx \neq Sy$, $Sy \neq Tx$ respectively.
If the function $x \to d(STx,Tx)$ attains its infimum on X or
the function $x \to \rho(TRx,Rx)$ attains its infimum on X or
the function $x \to \sigma(RSx,Sx)$ attains its infimum on X then S or T or R has a fixed point.

PROOF :

Assume that $x \to d(STx,Tx)$ attains its infimum on X. Then there exists $z \in X$ such that $d(STz,Tz) = \text{Inf}\ \{d(STx,Tx) / x \in X\}$

$$= f(z) \qquad \text{(say)}$$

Suppose R, S and T have no fixed points in X.

From (6.1) we have

$$f(RSTz) = d(STRSTz, TRSTz)$$
$$< \max\{\rho(TRSTz, RSTz), d(STRSTz, TRSTz), \rho(TRSTz, RSTz), \sigma(TRSTz, TRSTz)\}$$
$$= \rho(TRSTz, RSTz) \quad \text{since } T \text{ has no fixed point.}$$

From (6.2) we have

$$\rho(TRSTz, RSTz) < \max\ \{\sigma(RSTz, STz), \rho(TRSTz, RSTz), \sigma(RSTz, STz), d(RSTz, RSTz)\}$$
$$= \sigma(RSTz, STz) \text{ since } R \text{ has no fixed point.}$$

From (6.3) we have

$$\sigma(RSTz, STz) < \max\ \{d(STz, Tz), \sigma(RSTz, STz), d(STz, Tz), \rho(STz, STz)\}$$
$$= d(STz, Tz) \quad \text{since } S \text{ has no fixed point.}$$
$$= f(z)$$

Thus we have $f(RSTz) < f(z)$, a contradicion.

Hence R or S or T has a fixed point.

Finally we prove the following Theorem for commuting maps.

THEOREM 7 :

Let X be a non-empty set ,'d','ρ','σ be non-negative functions on $X \times X$ such that $d(x_1, x_2) = 0$ iff $x_1 = x_2$, $\rho(x_1 x_2) = 0$ iff $x_1 = x_2$ and $\sigma(x_1 x_2) = 0$ if $x_1 = x_2 = 0$ iff $x_1 = x_2$. Let R, S, T : $X \to X$ be commuting pair wise such that

$$d(RSy, RSTx) < \rho(y, Tx) \text{ for } y \neq Tx, RSy \neq RSTx \tag{7.1}$$

$$\rho(TRz, TRSy) < \sigma(z, Sy) \text{ for z } \neq Sy, TRz \neq TRSy \tag{7.2}$$

$$\sigma(STx, STRz) < d(x, Rz) \text{ for } x \neq Rz, STx \neq STRz. \tag{7.3}$$

Then 'R' has a fixed point if the function $x \to d(x, Rx)$ attains its infimum on X,

$'S'$ has a fixed point if the function $x \to \sigma(x, Sx)$ attains its infimum on Y and $'T'$ has a fixed point if the function $x \to \rho(x, Tx)$ attains its infimum on Z.

PROOF :

Assume the function $x \to d(x, Rx)$ attains infimum at $z \in X$.

That is $d(z, Rz) = \text{Inf}\,\{d(x, Rx) / x \in X\} = f(z)$ (say.)

Suppose $'R'$ has no fixed point. Then

$f(RSTRSTz) = d(RSTRSTz, RRSTRSTz)$

$= d(RSTRSTz, RSTRSTRz)$ (since R, S, T are commuting in pairs)

$< \rho(TRSTz, TRSTRz)$ from (7.1) since $R\ (TRSTz) \neq TRST\,z$

$< \sigma(STz, STRz)$ from (7.2) since $R\ (STz) \neq STz$.

$< d\ (z,\ Rz)$ from (7.3) since $Rz \neq z$.

$= f(z)$, a contradiction.

Therefore R has a fixed point.

Similarly we can prove the remaining cases.

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ANTIFUNGAL ACTIVITY OF CERTAIN PLANTS AGAINST *ALTERNARIA BRASSICAE* AND *ASPERGILLUS NIGER*

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ABSTRACT

The present paper deals with the degree of antifungal activity of four types of aqueous extracts of 30 plant species from Kumaon, Garhwal and Tarai regions against two fungal organisms, *Alternaria brassicae and Aspergillus niger*. Results revealed that aqueous extracts of eight plants were effective against *A. brassicae* alone but only two of them, *Boenninghausenia albiflora and Cannabis sativa*, were found effective against *A. niger* alone. The extracts of five plants, *Aegel marmelos, B. albiflora, Cleome viscosa, Leonotis nepetaefolia, and Vitex negundo* have been found to show antifungal activity against both of the fungli. However, the aqueous extracts of fifteen plants could show a low degree of antifungal activity. The specific and common inhibitory activities of plant extracts against these fungi might be interpreted due to presence of selective or wide-spectrum chemical compounds present in the plants.

Keywords: Thirty plants, Aqueous fresh/ dry and cold/hot extracts, Antifungal activity, *Alternaria brassicae* and *Aspergillus niger*.

INTRODUCTION

Albeit a large number of plant diseases have been controlled by using synthetic chemicals such as fungicides, pesticides and herbicides since long time, still it is in practice. Voluminous informations are available in this regard and some of the important contributions can be quoted [3, 5-6, 9, 19, 29]. Due to adverse effect of these chemicals in environment and their hard resistance for biodegradation, researchers have opened a new application of plants as biological agents for controlling the pests and plant pathogens that has taken an important place in pest and plant diseases' management programme since last few decades in India and abroad because they do not have such adverse effect on biological and physical system of our environment. These may contain many active ingredients although in low concentration. Degree of antifungal, antibacterial and antiviral property depends on nature and amount of the chemical substances produced by the plant system and stored form time to time in the different plant parts and their solubility at the time of their application. The pests and pathogens do not develop resistance to them, because it might require several simultaneous mutations to occur in the genetic constituents

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of the pests and pathogens in order to overcome the ingredients of the botanical pesticides [6, 24]. Plants are used for control of various bacterial, fungal and viral diseases of plants [6, 8, 10-12, 20-21, 24-25]. The chemical characterization of many of these plants has already been done [7, 11, 16, 26-27]. Systematic studies on the role of biocidal compounds from these plants for crop protection have not yet been made due to the lack of proper screening formats and indicator organisms of the plant diseases.

In the present study, 35 plant spp. suspected to possess bioactive chemicals were selected on the basis of information available in literature [7, 18, 20, 26-27], local folklore and field observations on plants that remain relatively free from the diseases. They are present at different height and locations in Kumaon, Garhwal, and Tarai region. However, only 30 plant spp. could be located and used for screening their fungicidal potential against the fungi, *A. brassicae and A. niger* which cause the leaf spot disease in brassicae and aflatoxin toxicity in maize and groundnut respectively. Though some of these plants have already been reported to show fungicidal action against *A. brassicae* and *A. niger* in the form of methanol extracts [4], they have not been screened for their activity in the form of aqueous extracts against the same fungi [18, 20, 27]. Many of these plants have been explored for their antibacterial activity recently [17].

MATERIALS AND METHODS

Collection of plants: All 30 plant spp. included in the present investigations have already been reported to exhibit medicinal / pesticidal or other economic values. These were collected from various places in Kumaon, Garhwal and Tarai region at latitude range 29^0 01' 20.0"' to 29^0 47' 52.1", longitude range 79^0 26' 41.9" to 79^0 35' 05.8" and altitude range 228 m to 2230 m. Different parts of the plants used for investigations are given in the table-1. The collected material was carried in polythene bags to the laboratory and dried in shade for 22 days.

Collection of fungal cultures: Fungal cultures of A. brassicae and A. niger were isolated from the hosts, brassicae and maize and tested for Kotch postulate at Department of Plant pathology, College of Agriculture, G.B. Pant university of Agriculture and Technology, Pantnagar, India from where these were procured by us. They were cultured in our laboratory in petriplates containing Patato Dextrose Ager (PDA) media.

Preparation of plant extract: 10 g of plant material collected from natural habitat was extracted four times with 2 ml. solvent/g of plant material at room temperature for preparation of fresh cold water extract (FCW) and in boiling conditions (temperature about 100^{0}C) for the fresh hot water extract (FHW). Remaining plant material was dried on shade. Dry powder of leaves/other parts of plant spp. was extracted four times with 5 ml. solvent/g of plant material for 48 hrs at room temperature or 100^{0}C for preparation of DCW and DHW respectively. All these extracts were combined and concentrated by flash evaporation at 40^{0}C to 2 ml (i.e. 20 μl extract equivalent to 100 mg of plant material). five factors were involved in the treatment i.e. (a) Fresh cold water extract (FCW) (b) Fresh hot water extract (FHW) (c) Dry cold water extract (DHW) (d) Dry hot water extract (DHW) and (e) two fungal pathogens, *A. brassicae and A. niger.*

Screening of extracts for antifungal activity: Paper disc method was used for screening anti-fungal activity of plant extract against test fungi. This method was based on diffusion capacity of test chemical (s) through an agar medium. Fungal plugs were placed at the centre of PDA plates and allowed to grow. After circular growth of about 2-3 cm diameter, four sterilized paper discs (two loaded with 20μl plant extracts and two with same amount of pure solvent) were placed at equal distance from center in order to see the antifungal potential of test plants against test fungi. These plates were incubated at 28 ± 1^{0}C for both the fungal pathogens. Inhibition zones were measured after four days of incubation. Dumb bell shaped growth of fungi was observed in case of extracts containing bioactive components. All the tests were carried out in triplicate and the mean values were used for interpretation.

RESULTS AND DISCUSSION

The extracts of *Acorus calamus* L., *Artemisia nilagirica* (Clarke) Pamp., *Cassia fistula* L., *Cassia occidentalis* L., *Ficus religiosa* L., *Hedychium spicatum* Smith., *Iris Kumaonesis* Wall., *Justicia adhatoda* L., *Lantana camara* L., *Litsea glutinosa* (Lour.) Robins., *Polygonum lapathifolium* L., *Sapium insigne* (Royle) Benth. ex. Hook. f., *Smilex aspera* L., *Urtica dioica* L. and *Viola biflora* L. were found ineffective against both of the fungal pathogens.

The results of fungicidal activity of other fifteen plants are presented in the table-1. *A. niger* is more resistant towards all plant extracts except those of *Cannabis sativa* L. and *Toona ciliata* Roem., the former being more potent (inhibition zone of

2-3 mm) in comparison to the latter (inhibition size 1mm. only). *Berberis aristata* DC., *Conyza bonariensis* (L.) Crong., *Erigeron karvinskianus* DC., *Lepidium apetalum* Willd., *Lyonia ovalifolia* (Wall.) Drude., *Mentha longifolia* (L.) Hudson., *Thalictrum foliolosum* DC., and *Woodfordia fruticosa* (L.) Kurz. are potent against *A. brassicae* only. While all of them show inhibation zone between 1-3 mm., *W. fruticosa* is most effective with zone size of 5-6 mm. The plant extracts of *Aegle marmelos* (L.) Corr., *Boenninghausenia albiflora* (Hook.) Reichb., *Cleome viscosa* L., *Leonotis nepetaefolia* (L.) W. Ait., and *Vitex negundo* L. are effective against both of the fungi involved in the present investigation with inhibition between 1-6 mm. However, *C. viscosa* and *V. negundo* extracts are better in fungidal potential against *A. brassicae* while other plant spp. are more potent against both of the test fungi.

A perusal of the data indicates that the solubility of the active constituents of extracts of some of the plant spp. are not influenced and no degradation takes place on heating up to 100^{0}C while in some cases, these parameters are expected to be enhanced or suppressed resulting in the increased or decreased fungicidal action respectively.

The reduced activity (when the hot extracts are used in some of the cases), may be due to the decreased solunility and degradation/deterioration of the active fungicidal ingredients of the plants used. Basic principles of the chemical energetics do not rule out the chances of reduction of solubility of certain chemical substances on elevating the temperature when the molar enthalpy of solubilization bears a negative value. Similarly, the increased antifungal activity on using hot aqueous extracts may be due to the increased solubility and activation of the active principles of plants involved. Increase in the fungicidal potential in some of the plants on using DHW in place of FHW implies the better exposure of the active componects on drying of plant material.

The fungicidal action of the extracts of *V. negundo*, *C. viscosa* and *W. fruticosa* against *A. brassicae* is advesely affected on using the hot extracts while it is the reverse case or no change is observed against *A. niger*. It points towards the possibility that different active principles of the same plant are acting as fungicidal agents against *A. brassicae* and *A. niger*. Actually, these plant spp. have shown the best activity against *A. brassicae* amongst all the 30 plants used for this study.

It is encouraging to find that the ornamental plants like *C. karvinskianus* and

E. bonariensis [13] can be used as fungicidal agents. Further, these are far better in fungicidal potential especially against *A. brassicae* in comparison to the plants *A. calamus*, *L. ovalifolia* and *T. ciliata* which are better known for their fungicidal other fungi [6, 2, 24] or insecticidal plants like *C. occidentalis*, *L. glutinosa* and *S. insigne* [8, 27] or bactericidal plant like *L. nepetaefolia* [10] or *P. lapathifolium*- a pesticidal plant [25]. Their fungicidal action is better than the medicinal plants like *A. marmelos*, *A. nilagirica*, *B. albiflora*, *C. fistula*, *F. religiosa*, *J. adhatoda*, *T. foliolosum*, *U. dioica* and *V. biflora* [18, 22-23, 26-28] which, in turn, are either not effective at all or have comparatively lesser fungicidal potential against the fungi used by us.

Many of these plants have been reported to exhibit antifungal activity when their aqueous/ methanot extracts were used against the fungal pathogens of rice like *Magnaporthe grisea* and *Rhizoctonia solani* [14-15] or against mushroom fungal pathogens like *Mycogone perniciosa* and *Verticillium fungicola* [1] or against the soybeen fungal pathogens like *Colletotrichum truncatum*, *Fusarium oxysporum* and *Macrophomina phaseolina* [2].

We have recently reported the fungicidal potential of the methanol extracts of dry powder of the leaves of these and some other plants against *A. brassicae* and *A. niger* [4]. A comparison with the repoted work [4] indicates that the degree of antifungal activity is not as good in aqueous extracts (both DCW and DHW) as in the case when the dry methanol extracts are applied. However, it is worth mentioning that water (and not the methanol), is readily available as a solvent in the natural system. Therefore, the active antifungal principles of plants are expected to be used in the form of aqueous extracts in the natural ecosystems. It makes the aqueous extracts more significant as the careers for the natural fungicides.

The effective plant extracts are expected to contain wide spectrum antifugal chemical (s). These plant species may contain chemical compound (s) having selective/ specific antifungal properties. Therefore, the results described above may be helpful in developing/ synthesizing the plant based natural fungicides that may be used for preventing the incidence of leaf spot of brassicae and aflatoxin toxicity in maize, groundnut etc.

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Table-1 : Inhibitory Effect of aqueous extracts of plants against *A. brassicae* and *A. niger*.

S. No.	Plant species (Part used)	Type * of Extract	Zone size (mm)	
			A. brassicae	*A. niger*
1.	*A. marmelos* (Leaf)	FCW	2	0
		FHW	2	4
		DCW	3	2
		DHW	2	6
2.	*B. aristata* (Leaf)	FCW	2	0
		FHW	2	0
		DCW	3	0
		DHW	3	0
3.	*B. albiflora* (Whole plant)	FCW	0	1
		FHW	0	1
4.	*C. sativa* (Leaf)	FCW	0	2
		DHW	0	3
5.	*C. bonariensis* (Leaf)	FCW	1	0
		FHW	0	0
		DCW	2	0
		DHW	2	0
6.	*C. viscosa* (Leaf)	FCW	2	2
		FHW	1	0
		DCW	6	5
		DHW	6	5
7.	*E. karvinskianus* (Leaf)	DCW	3	0
		DHW	2	0
8.	*L. nepetaefoha* (Leaf)	FCW	0	2
		FHW	0	2
		DCW	1	2
		DHW	1	3
9.	*L. apetalum* (Leaf)	FCW	0	0
		FHW	0	0
		DCW	1	0
		DHW	2	0
10.	*I. ovalifolia* (Leaf)	FCW	1	0
		DCW	2	0
11.	*M. longifolia* (Leaf)	DCW	1	0
		DHW	1	0
12.	*T. foliolosum* (Bark)	FCW	1	0
		FHW	2	0
		DCW	2	0
		DHW	2	0
13.	*T. ciliata* (Bark)	FCW	0	1
		FHW	0	1
14.	*V. negundo* (Leaf)	FCW	5	2
		FHW	4	3
		DCW	6	4
		DHW	5	4
15.	*W. fruticosa* (Leaf)	FCW	6	0
		FHW	5	0

*FCW – Fresh cold aqueous extract ; FHW – Fresh hot aqueous extract
DCW – Dry cold aqueous extract ; DHW –Dry hot aqueous extract

of the fungal cultures and in identification / collection of plant species respectively.

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SOME GEOMETRICAL CONSTRUCTIONS FROM BAUDHĀYANA ŚULBA SŪTRA

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ABSTRACT

Some geometrical constructions from Baudhāyana Śulba Sūtra are given.

Key words and Phrases : Square, Side, Tie, Pole, Rectangle, Oblong, Circle, Triangle, Prauga, Srori, Cord, Prācī and Area etc.

Classification Number : 01A32

INTRODUCTION

Śulba Sūtra-

The science of mathematics which is said to be the "queen of all sciences" or "the peak of all sciences" has orginated in so hoary manner that it is impossible to trace its origin in any of the written records of the human civilization. Mathematics was known in ancient India as Gaṇir-veda or the Veda (i.e. knowledge) of calculations See 6 & 7.

India's oldest written works are the Vedas. There are four Vedas, namely Ṛgveda (ऋग्वेद) which is the oldest, Yajurveda, Sāmaveda, and Atharvaveda also, associated with a Veda. There are different branches or schools which are represented by the various Saṁhitās (or recensions) of that Veda. To assist the proper study of the Veda, there are Six Vedāṅgas, ("limbs or parts of the Veda"). Namely Śiksā (Phonetics), Kalpa (Ritualistics), Vyākaraṇa (Grammar), Nirukta (Etymology), Chands (Prosody), and Jyotiṣa (Astronomy and Astrology), as is clear from the popular verse See 1,3 & 5.

शिक्षा कल्पो व्याकरणं निरूक्तं छन्दसांचयः।
ज्योतिषामयनं चैव वेदाङ्गानि षडेव तु।।

The Kalpa deals with rules and methods for pertorming vedic rituals, sacrifices and ceremonies, and is divided into three categories which are called Śrauta (श्रौत), Gṛhya (गृह्य) and Dharma (धर्म) See 5.

The Śrauta Sūtras, especially those belonging to the various saṁhitās of the Yajurveda. Often include tracts which give rules concerning the mensuration and construction of vedis

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(sacrificial grounds), citis (mounds or altars), and agnis (fire-places). Such tracts are also found as separate works and are called Śulba Sūtras (शुल्ब सूत्राणि) (which are denoted by SS here after). Thus SS are the oldest geometrical treatises which represent, in coded form, the much older and traditional Indian mathematics developed for construction and transformation of vedic altars of various types and forms. The word śulba literally means a cord, rope, or string and is derived from the basic root śulb (or śulv) meaning "to mete out" or "to measure" See 5.

By now the names of about a dozen SS are known. They are Baudhāyana, Āpastamba, Kātyāyana, Mānava, Satyāṣāḍha, Maitrāyana, Varāha, Vadhula, Maśaka, Hiraṇyakeśin, and Laugākgi. They are variously dated their exact times of composition or compilation is controversial. The Baudhāyana Śulba Sūtra (=BSS) which is the oldest of them is generally placed between 800 B.C. and 500 B.C. The Āpastamba Śulba Sūtra (=ASS), Kātyāyana Śulba Sūtra (=KSS), and Mānava Śulba Sūtra (=MSS) are other important ancient works of the class. The text of Satyāṣāḍha is said to be identical with that of ASS while Maitrāyaṇa is said to be another recension of the MSS See 5.

SOME GEOMETRICAL CONSTRUCTIONS

1. TO CONSTRUCT A SQUARE WITH THE HELP OF A CORD ALONE

चतुरस्त्रं चिकीर्षन्यावच्चिकीर्षेत्तावीं रज्जुमुभयतः पाशां कृत्वा मध्ये लक्षणं करोति लेखामालिख्य ।।
(BSS. 1/22)

If you wish to make a square, take a cord of the length which you desire to give to the side of the square, make a tie at both its ends and a mark at its middle; then having drawn the line (i.e., praci, the line pointing to the east and west points; the different methods by which this line was found will be discussed later.)

तस्या मध्ये शङ्कुं निहन्यात्तस्मिन्पाशौ प्रतिमुच्य लक्षणेन मण्डलं परिलिखेत् विश्कम्भान्तयोः शङ्कू निहन्यात् ।। (BSS. 1/23)

Fix a pole in the middle of that line; having fastened at that pole the to ties of the cord, describe (around the pole) a circle with the mark. (As the mark divides the length of the cord, the circle is to be describe with half the length of the cord); fix poles at both ends of the diameter (i.e., diameter formed by the prācī).

पूर्वस्मिन्पाशं प्रतिमुच्य पाशेन मण्डलं परिलिखेत् ।। (BSS. 1/24)

Having fastened one tie at the eastern pole (the pole fix at the east and of the prācī) described a circle with the other tie (i.e., with the full length of the cord).

एवमपरस्मिऽस्ते यत्र समेयातां तेन द्वितीयं विष्कम्भमायच्छेत् ।। (BSS. 1/25)

In the same manner, a circle describe round the pole at the west end of the prācī; then a second diameter is drawn joining the points in which these two circles intersect (this diameter is the line pointing to the north and south points).

विष्कम्भान्तयोः शङ्कू निहन्यात् ।। (BSS. 1/26)

Fix a pole at both ends of the diameter.

पूर्वस्मिन्पाशौ प्रतिमुच्य लक्षणेन मण्डलं परिलिखेत् ।। (BSS. 1/27)

Having fastened both ties at the east pole, describe a circle round it with the mark.

एवं दक्षिणात एवं पश्चादेवमुत्तरतस्तेषां येऽन्त्याः संसर्गास्तच्चतुरस्त्रं संपद्यते ।। (BSS. 1/28)

The same is to be done in the south, the west, and the north (i.e., circles are to be described round the three other poles); the points of intersection of these four circles which (i.e., the points) are situated in the four intermediate regions (north-east, north-west etc.) are the four corners of the required square.

2. TO CONSTRUCT A SQUARE EQUAL TO THE SUM OF DIFFERENT SQUARES

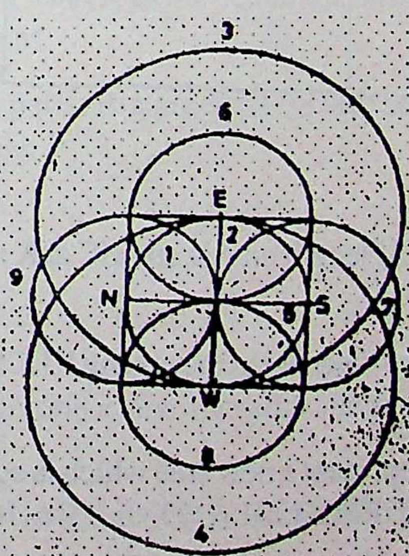
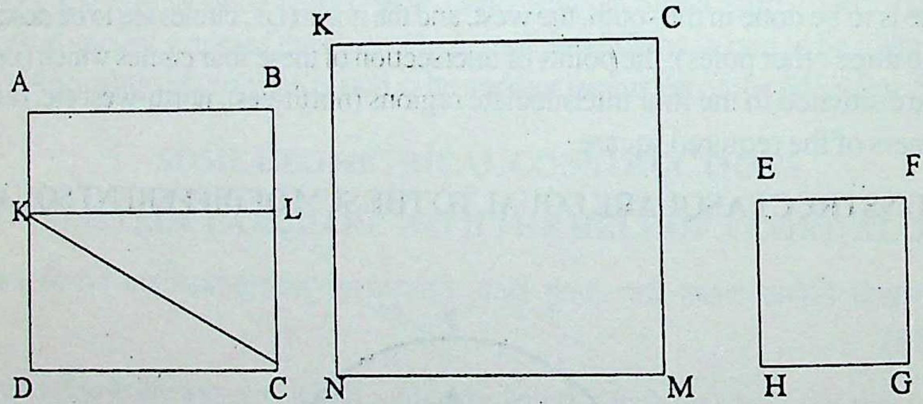


Fig. 1 : (Adhyaya 1, Sutra 28)

नाना चतुरस्त्रे समस्यन्कनीयसः करण्या वर्षीयसो वृध्रमुल्लिखेद्वृध्रस्याक्ष्णायाज्जुः समस्तयोः पार्श्वमानी भवति।। (BSS. 1/50)

If you wish to combine to squares of different Size into one, scratch up with the side of the smaller square a piece cut-off from the large one (i.e., cut-off a piece from the large square by scratching up the ground-or making a mark upon the ground-at a distance from one end of a side of the large square, which is equal to the length of the side of the smaller square; by repeating this process on the opposite side of the large square and joining the two marks on the ground by a line or cord, and oblong is cut-off, of which the two longer to sides are equal to the side of the large square and the two shorter sides to the side of the small square). The diagonal of this cut-off piece is the side of the combined squares (of the square which combines the two squares).



ABCD is a big square. Cut EH length on AD and BC at K and L respectively. Join KL and KC.

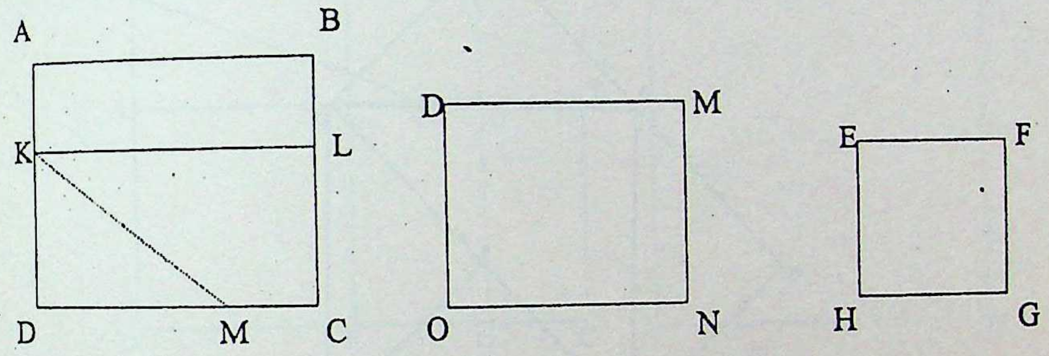
An oblong KLCD is cut of, the diagonal of the oblong KLCD is the side of the combined squares.

Hence square KCMN will represent a new area which is equal to the sum of area of the sq. ABCD and area of the sq. EFGH.

3. TO CONSTRUCT A SQUARE EQUAL TO THE DIFFERENCE OF TWO SQUARES

चतुरस्त्राच्चतुरस्त्रं निर्जिहीर्षन्यावनिनर्जिहीर्शेत्तस्य करण्या वर्षीयसो वृध्रमुल्लिखेत्। वृध्रस्य पार्श्वमानीमक्षणयेतरत्पार्श्वमुपसंहरेत्सा यत्र निपतेतदपच्छिन्द्याच्छिन्नया निरस्तम्।। (BSS. 1/51)

If you wish to deduct one square from another square, cut-off a piece from the large square by making a mark on the ground with side of the smaller square which you wish to deduct (an oblong is cut-off, the sides of which are equal to the sides of the two given squares); draw one of the sides (the cord representing one of the longer sides of the oblong) across the oblong so that it touches the other side; where it touches (the other side), by this line which has been cut-off the small square is deducted from the large one (i.e., the cut-off the line is the side of a square the area of which is equal to the difference of two squares.)



ABCD is a big square. Cut EH length on AD and BC at K and L respectively. Join KL cut the length KL at the DC taking form K as the center put an are equal to KL on the base DC. Let we name this point as M.

Now base is divided into two parts. Big portion DM will be side of a required square because in a right angled triangle.

$KD^2 + DE^2 = KM^2$

$EF^2 + DM^2 = AB^2$

or $DM^2 = AB^2 - EF^2$

Hence square DMNO will represent a new area after subtracting area of the sq. EFGH from sq. ABCD.

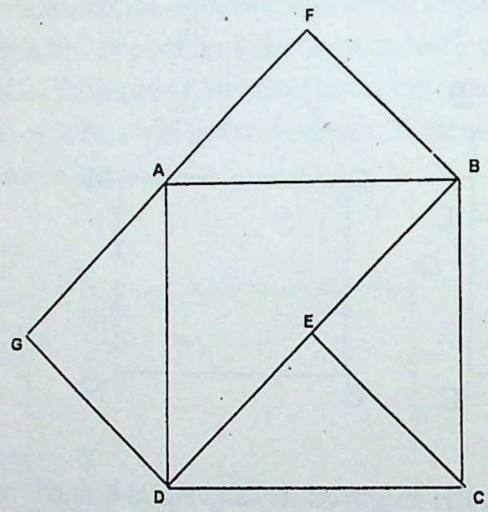
4. TRANSFORMING A SQUARE INTO A RECTANGLE

समचतुरस्रं दीर्घचतुरस्रं चिकीर्षꣳस्तदक्ष्णायापच्छिद्य भागं द्वेधा विभज्य पार्श्वयोरुपदध्याद्यथायोगम् ।।
(BSS.1/52)

To convert a square into rectangle Baudhāyana suggests that divide a square into

two parts by a diagonal. Again divide one part of Square into two parts. Thus we get two isosceles triangles.

Now place these two isosceles triangles on the opposite corners of the adjacent sides i.e. one on northern side and other on eastern side so that one rectangle is formed See 3.



CONSTRUCTION AND PROOF :

Draw a square ABCD. Join BD to divide it into two parts so that triangle DAB and triangle DCB are obtained. Again divide triangle DBC into two parts such that two isosceles triangles are obtained triangle EDC and triangle ECB. Place triangle ECB on triangle FAB (eastern side of square ABCD) such that C on A, B on B, E on F. Place triangle EDC on triangle GDA such that D on D, C on A, E on G. Thus GDBF is a required rectangle (see 4).

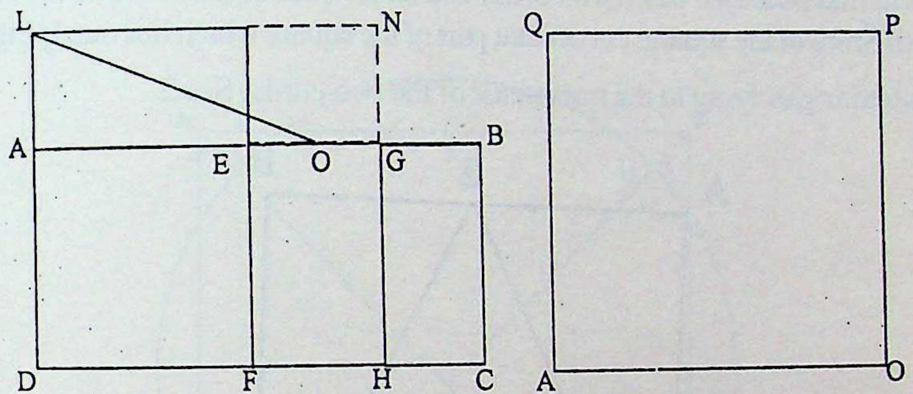
5. METHOD OF CONVERTING A RECTANGLE IN A SQUARE

दीर्घचतुरस्रं समचतुरस्रं चिकीर्षन्स्तिर्यङ्‌मानी करणीं कृत्वा शेषं द्वेधा विभज्य पार्श्वयोरूपदध्यात् खंडमावापेन तत्संपूरयेन्तस्य निर्हार उक्तः।। BSS.1/54)

To transform ABCD a rectangle into square, draw a line EF bisecting the lines AB & DC. In this way AEFD and EBCF two squares are formed. Now in square EBCF, draw a line GH bisecting sides EB & FC. Therefore EBCF is divided into two rectangles

GBCH and EGFF. Place GBCH at AE and we have LMEA. Now complete square by placing small square MNGE above EGHF at corner. Thus LNHD is a big square. To convert it into a square whose area is equal to area of rectangle ABCD, we have to separate only the area of small square MNGE.

Take L as centre cut at AG equal to LN. Let it cut AG at O, complete a square with side AO. Hence AOPQ is a required square.



METHOD OF CONSTRUCTION AND PROOF :

To transform a rectangle into a square Baudhāyana suggests this following method.

ABCD is a rectangle whose side BC=½ AB (i.e. breadth is half of the length). Divide rectangle divided into two equal parts. Now other half is again divided into two parts. Its one half is GBCH which is eliminated from south and is placed on the east side of first half square AEFD. Now complete a square by introducing a small square at the corner so that LNHD is obtained. Now a/c to given condition

Area of rectangle ABCD = area of Sq. LNHD-area of Sq. MNGE.

Now with centre L and radius equal to LN Cut off AG at O. Join LO. The square on AO is equal to the difference between two squares.

$$AO^2 = OL^2 - AL^2$$

$$AL^2 + AO^2 = OL^2 \qquad \text{and } OL = LN$$

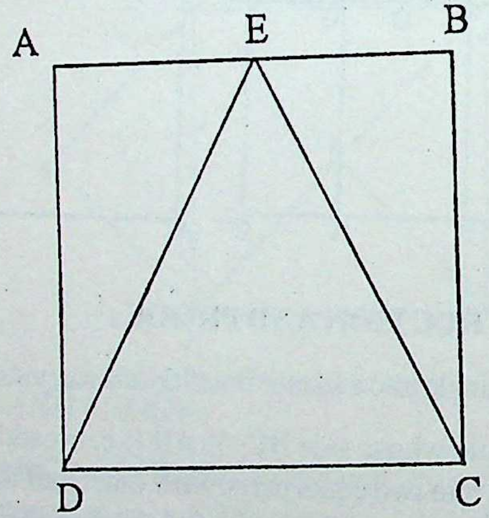
$$AO^2 = LN^2 - AL^2$$

Therefore, draw a square whose side is AO. Hence, new square whose area will be equal to rectangle ABCD is QPOA See 4.

6. TRANSFORMATION A SQUARE INTO A TRIANGLE

चतुरस्रं प्रउगं चिकीर्षन्यावच्चिकीर्षेद् द्विस्तावतीं भूमि समचुतुरस्त्रां कृत्वा पूर्वस्याः करण्या मध्ये शङ्कुं निहन्यात्तस्मिन्पाशौ प्रतिमुच्य दक्षिणोत्तरयोः श्रोण्योर्निपातयेद्वहिः स्पन्द्यमपच्छिनद्यात्।। (BSS.1/56)

It you wish to turn a square into a triangle, make a square the area of which is twice as large as the area of the triangle you wish to make. Fix a pole in the middle of the eastern side, faster at that pole two ties (two cords) and stretch the cords towards the southern and northern Śroni of the square; cut off that part of the square which lies outside the cords (i.e., the two triangles lying to the north-east of the two cords) See 2.



CONSTRUCTION AND PROOF :

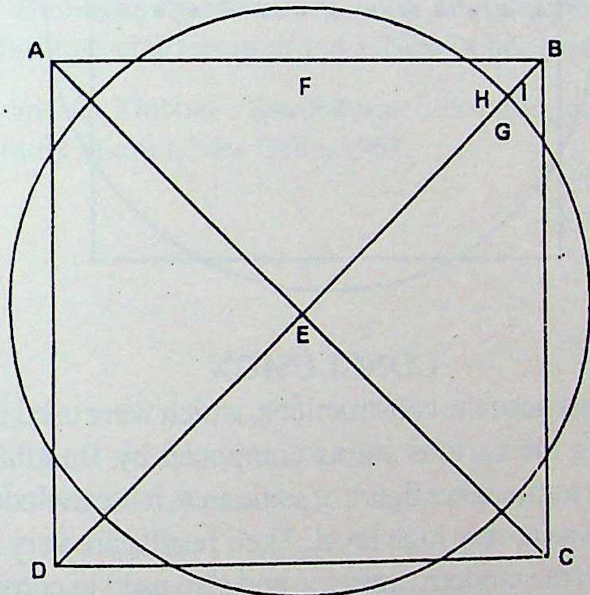
Draw a square ABCD. 'E' is the middle point of the side AB. Join ED and EC, cut off part ECD of the square which lies between the two triangles EDA and ECB.

Hence, ECD is a required triangle whose area will be half of the square ABCD See 4.

7. TRANSFORMATION A SQUARE INTO A CIRCLE

चतुरस्र मंडलं चिकीर्षन्नक्ष्णायार्ध मध्यात्प्राचीमम्यापात यद्यदति–शिष्यते तस्य सह तृतीयेन मंडलं परिलिखेत्।।। (BSS.1/58)

If you wish to turn a square into a circle, draw half of the cord stretched in the diagonal from the centre towards the prācī-line (i.e., stretch a cord from the centre of the square to one of the corners, for instance to the north-cast corner and move then the loose end of the cord towards south until the cord covers the prācī the line running from the centre of the eastern side of the square to the centre of the western side; a piece of the cord will them, of course, lie outside the square), describe the circle together with the third part of that piece of the cord which stands over (i.e., take for radius of the circle the whole piece of the cord which lies inside the square plus the third part of the piece which lies outside) See 2.



CONSTRUCTION AND PROOF :

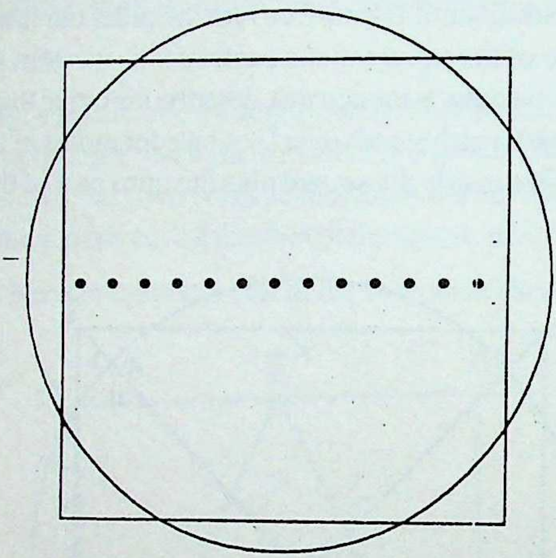
Transtorm ABCD square into a circle. Draw diagonal AC, BD from angle A and angle D. E is their meeting point, cut half of AB at F, cut off arc EG=FB from E. Divide GB into three parts GH, HI, IB. Now leave two parts and with centre EH draw a circle whose area will be equale to that square See 4.

8. TRANSFORMATION A CIRCLE INTO A SQUARE

मंडलं चतुरस्रं चिकीर्षन्विष्कम्भमष्टौ भागान्कृत्वा भागमेकोनत्रिंशधा विभज्याष्टाविंशतिभागानुद्धरेद् भागस्य च षष्ठमष्टमभागोनम्।। (BSS.1/59)

If you wish to transform a circle into a square, divide its diameter into 8 parts and then divide one of these parts into 29 parts and subtract 28 of these and also the sixth part (of the proceeding sub-division) less the eighth part (of last).

The meaning is : 7/8+1/8.29-1/8.29.6+1/8.29.6.8 of the diameter of a circle is the side of a square the area of which is equal to the area of the circle See 2.



CONCLUSION

Given above are some of the constructions, which were used in the preparation of altars. After discussing the various sutras composed by Baudhāyan regarding the conversion of one figure into another figure of same area, it is concluded that in Vedic age, geometric knowledge was at very high level. Their results are very much applicable as mathematical devices in the modern times too and also used in compounding of land in terms adding and subtracting land. These sūtras show that there was a concept of altering the different shapes of land into a desired one.

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SIMILARITY OF PLANE FIGURES AND GEOMETRIC & GROUP THEORETIC STUDY OF CYCLIC QUADRILATERALS OF FIRST KIND

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ABSTRACT

Similarity of plane figures is a very old concept. The approach is to study the valuable contributions by the Arabs, Babylons/Summerians, Chinese and Greeks with reference to the Indians at large and importance of the concept as such. Finally, we study cyclic quadrilaterals of first kind geometrically. Its group theoretic study concludes with a non-abelian group of eight permutations in which four permutations are equal to another four.

Section A

INTRODUCTION

The concept of similar plane figures which is of ritual origin and whose reference goes back to at least the period of the *Śatapatha Brāhmaṇa* (ca. 2000-250 B.C.) originated due to the oftentimes necessity to construct, during the sacrificial rites, an altar similar to but differing in area from another by a specific amount. In it, for instance, regarding different layer construction of the *Caturasara-śyenacit*, a primitive fire altar, is said that it "should have an area of seven and a half square *puruṣa* at the time of first construction. At the second construction, its area shall have to be $8\frac{1}{2}$ square *puruṣas*, at the third $9\frac{1}{2}$ square *puruṣas*. In the same manner the area of the *Agni* should be increased by one square purusa at each successive construction up to $101\frac{1}{2}$ square *puruṣas*" (without depriving it (the *Agni*) its due proportions, i.e., shape) [1]. Construction of a *vedi* 14 or $14\frac{3}{7}$ times as large as the *Mahāvedi* and falcon-shaped *vedis* are the other examples to its credit (*op. cit*, pp. 158-160). An elaborated collections and the other details as regards the construction of similar figures pertaining to various altars are available in the manuals, called the *Śulbasūtras* (ca. 800-200 B.C.). In these, for example, "the *Sautrāmaṇikī -vedi* is stated to be equal to one third the *Mahāvedi* and the *vedi* of *Aśvameda* double the latter. The *Lokṣahoma-vedi* and *Kotihoma-vedi* are respectively four and twenty-five times the *Pakayajniki-vedi*" [2]. A set of similar transformations for circles, rectangles, rhomboids, trapezia, triangles etc. may be found in

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the *Śulbasūtras* thus arising the propositions, a few popular among them are discussed here.

1. To construct an isosceles trapezium similar to a given one but of n times the size of it. The principle underlying is to replace the unit of measurement of the latter by $\sqrt{n}$ times (*Ibid*, p. 154).
2. To construct a fire-altar similar to that of the shape of a falcon, but differing from its primitive area by m square unit (*Ibid*).
3. To construct a trapezium, provided one side (or altitude) of former is known (*Ibid*, p. 163).
4. To construct a rectangle similar to a given rectangle by successively varying the units (*Ibid*).

THEOREM OF SQUARE ON HALF CHORD

The *Áryabhātiya*(=*AB*) (c. 499 A. D.) of *Āryabhaṭa* regarding the above theorem reads as: In a cricle (when a chord divides it into two arcs), the product of the arrows of the two arcs is certainly equal to the square of half the chord [3]".

Mathematically speaking, if in a circle (Figure 1) a chord CD is bisected at E by a diameter AB perpendicular to it, then

(1) $AE. EB = CE^2$.

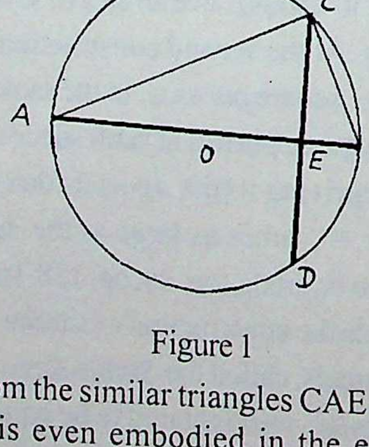

Figure 1

Its proof easily follows from the similar triangles CAE and CEB (see [4], p. 53). The flavour of this theorem is even embodied in the early Jaina works [5] as the *Jambūdvipasmāsa* and the *Tattvārthādhigama-sūtra* composed of Umasvati of first century A. D. atleast, a formal translation of latter which correlates the chord (c), the depth (h), and the diameter (d) of a circle runs as: "The square-root of four times the product of an arbitrary depth and the diameter diminished by that depth is the chord" (iii, 11). That is $c = \sqrt{4h(d-h)}$.

Trustworthy evidence of the late compositions [6] where the rule is said to be recurring in one form or the other include, for instance, *Bṛhatkṣetrasamāsa* of Jinabhadra Gaṇi (ca. 609 A. D.), the *Brāhmasphuṭasiddhānta* (= *BrSS*) (c. 628 A.D.) of Brahmagupta, the *Līlāvatī* (=*LV*) of *Bhāskara* II (b. 1114 A.D.), the *Naradapurāṇa* (=*NP*) (ca. twelfth century A.D.) and also *Gaṇita Kaumudī* of *Nārāyaṇa Paṇḍita* (ca. 1356 A.D.).

Here are a few interesting examples based on the rule (1).

Problems from Bhāskara I's commetary the *Āryabhaṭīyabhāṣya* (=*ABB*) (c. 629 A.D.) on the diameter *AB*

I. Broken bamboo problem

We cite here one of the two similar examples proposed by *Bhāskara* I.

A bamboo of height 18 is felled by the wind. It falls at (a distance of) 6 from the root (thus) forming a trriangle. Where is the break? [7]

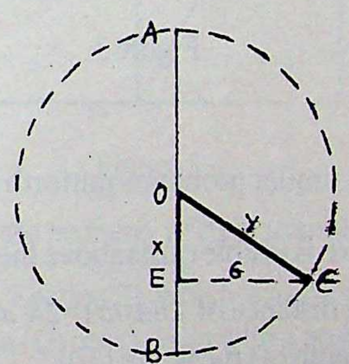

Figure 2

A few other references [8] of the similar problems are :

1. The *Gaṇitasārasaṅgraha* (=*GSS*) (c. 850 A.D.) of *Mahāvīra*
2. The commentary (c. 860 A.D.) on the *BrSS* by *Pṛthudakasvami*
3. The Bijaganita and the *LV* of *Bījagaṇita* and the *LV* of *Bhāskara* II
4. The commentary on the *AB* by *Someśvara* (fl. 916-1200 A.D.)
5. The *Gaṇitakaumudī* (=*GK*) (c. 1356 A.D.) of *Nārāyaṇa Paṇḍita*
6. The commentary on the *AB*(=*ABRR*) by *Raghunātha Rāja* (ca. 1597 A.D.)

Noticiable that the Chinese text *Chiu-chang suan-shu* (Nine Chapters on Mathematical Art, ca. 100 B.C.) too contains a problem similar to *Bhāskara* I in its ninth chapter entitled "kou ku" (Right Angled Triangles). Its solution is understood to be based on the Pythagorean property; Problem 13.

"A bamboo is 1 *chang* (= 10 *chhih*) tall. It is broken, and the top touches the ground 3 *chhih* from the root. What is the height of the break"? [9]

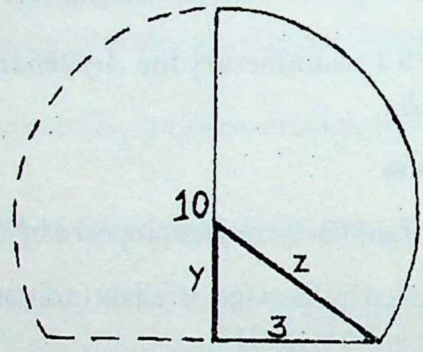

Figure 3

II. Lotus problem

Here is one of the two similar problems putforth by *Bhāskara* I.

"A full-blown lotus 8 *aṅgulas* is visible (just) above the water. When carried away by the wind, it submerges just at the distance of 1 *hasta* (=24 *aṅgulas*). Tell quickly (the height of) the lotus plant and (the depth of) the water [10]."

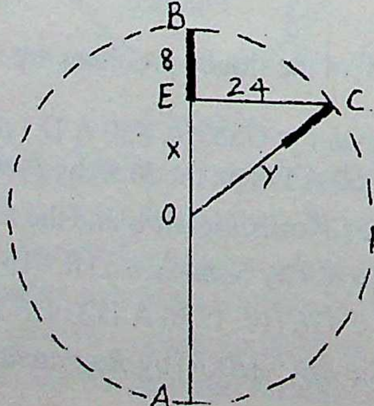

Figure 4

The similar problems [11] occur also in the *LV* and the *GK*.

Worth mentioning that one such problem similar to *Bhāskara* I we also find as the "problem of reed in a pond" in the ninth chapter of the *chiu-chang suan-shu*; Problem 6.

"There is a pond whose section is square of side 1 *chang* (= 10 *chhih*). A reed grows in its centre and extends 1 *chhih* above water. If the reed is fulled to the side (of the pond), it reaches the bank precisely. What are the depth of the water and the length of the reed [12]?"

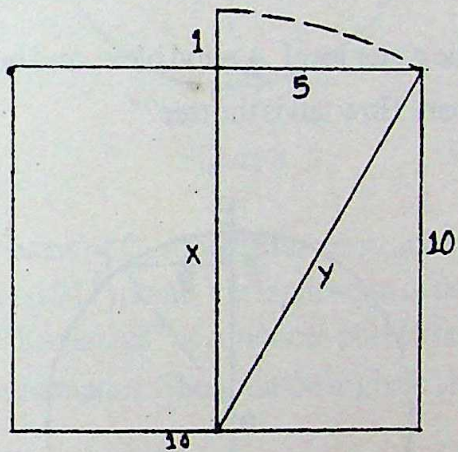

Figure 5

Certainly its solution is based on the property of right triangle.

In the Collection (Book III, Section 2), which is a commentary of Elements, Pappus (third century A.D.) of Alexandria (Greece) deals with a problem to show in a semi-circle the three means of arithmetic, geometric and harmonic. Following Heath [13], OC, BD & DF respectively stand for arithmetic mean, geometric mean and harmonic mean between two numbers AB and BC.

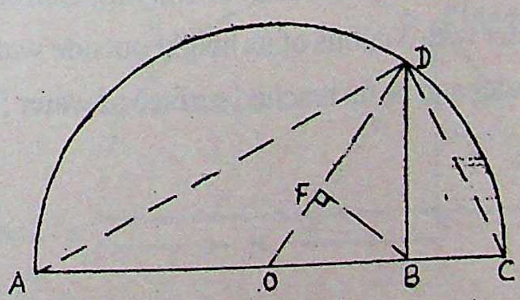

Figure 6

To mention here in this aspect that arithmetic and the theory of music led early Pythagorean School to develop the three means. The third mean now called harmonic due to Arahytas and Hippasus was then called 'subcontrary'; the other two were known as these are known presently [14].

Problem 42 [15] of the Persian arithmetic treatise *Miftāḥ al-muămalāt* ("Key to Transactions") by Tranian mathematician and astronomer Muhammad bin Ayyūb Ṭabarī of 11-12th century A.D. [Ed. M. -A. Rīyahī, Tehran, 1970, in Persian].

"A tree 3 cubits was outside water level. A wind blew, and bent it so that its top reached the water level, 9 cubits apart. How tall is the tree?"

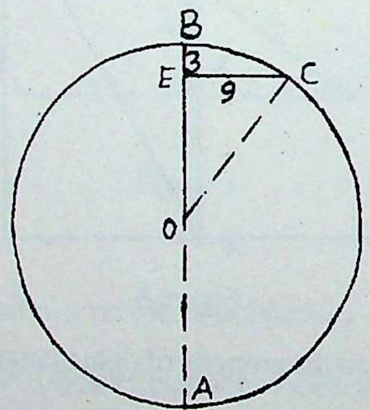

Figure 7

Ghiyath al-Din Jamshid al-kashi's Miftah al-hisab (first of the eight geometrical problems) [16] [Ed. A.S. Damardash and M.H. al-Hafni al Shaykh, Cairo, 1970].

"A spear is planted in water with 3 cubits of its height outside water. A wind inclined to it so that it submerged in water and its tip reached surface of water 5 cubits away. How long is the spear?"

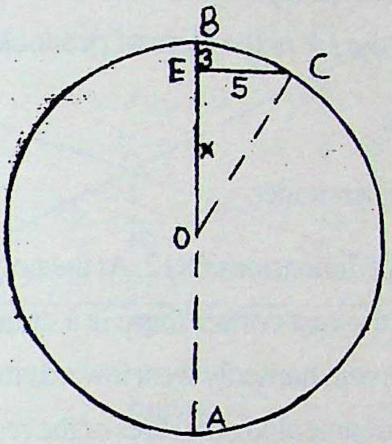

Figure 8

Bagheri [15] on the basis of Gardner [Mathematical Puzzles of Sam Lord, New York, 1959; Sam Lord 1841-1911] points the replication of the problem of *Bhāskara* II as 'Lily problem' in a novel "Kavanagh" of American poet Henry Wordsworth Longfellow (1807-1882), a lover of mathematics. The solution is given to be similar to Ṭabarī.

II. Hawk and rat problem

One of the two problems from *Bhāskara* I's commentary is :

"A hawk is sitting on a pole whose height is 18 (cubits). A rat, who has gone out of his dwelling (at the foot of the pole) to a distance of 81 (cubits), while returning towards his dwelling, afraid of the hawk, is killed by the cruel (bird) on the way. Say how far has he to go to reach his hole (if he were to alive), and also the (diagonal) motion of the hawk (the speeds of the rat and the hawk being the same) [17]."

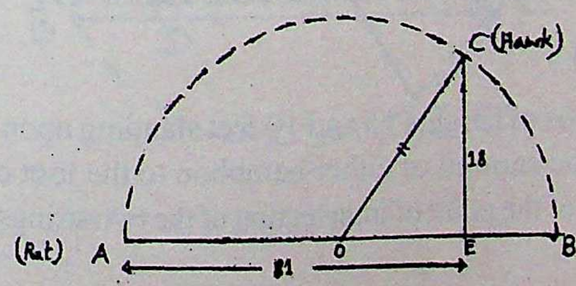

Figure 9

The same problem reappears in Yallay's super commentary [18] on the *AB*. The similar problem is stated to occur in the *LV* in the form of peacock and serpent pair [19].

IV. Crane and fish problem

The problem from the *ABB* is hereunder.

"There is a reservoir of water of dimensions 6X12. At the north-east corner of the reservoir there is a fish; and at the north-west corner there is a crane (*bake*). For the fear of the crane the fish, crossing the reservoir, hurriedly went towards the south in an oblique direction but was killed by the crane who came along the sides of the reservoir. Give out the distances travelled by them (assuming that their speeds are the same) [20]."

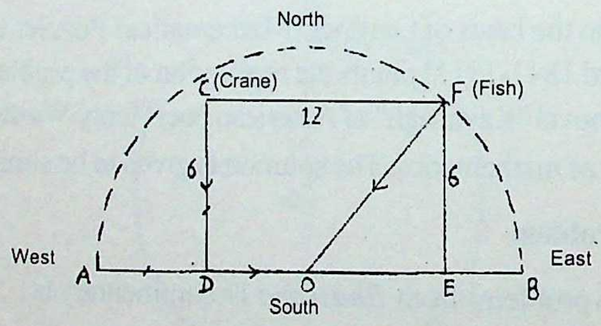

Figure 10

The same problem is claimed to be reproduced in the *GK* and the *ABRR* [21].

PROBLEM FROM THE *LĪLĀVATĪ*

Bumboo problem [22].

"There are two bamboos of heights 15 and 10 feet standing upon the ground and tight strings are tied from the summit of either bambboo to the foot of the other. Find the perpendicular distance of the point of intersection of the two strings from the ground."

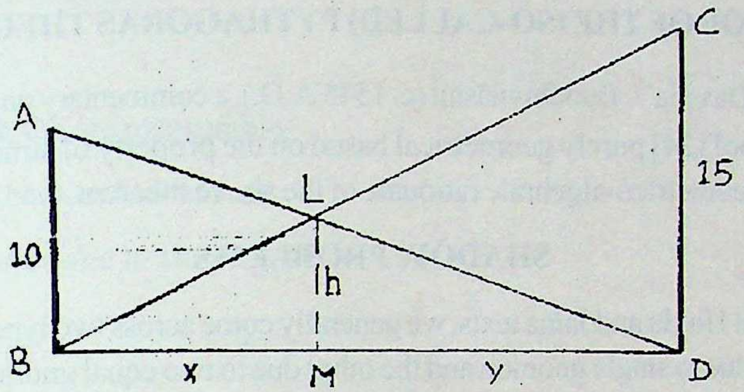

Figure 11

BRAHMAGUPTA'S THEOREM

Statement [23]

"The rectangle contained by the two sides of a triangle is equal to the rectangle contained by the circum-diameter and the altitude to the base." *BrSS* XII.27.

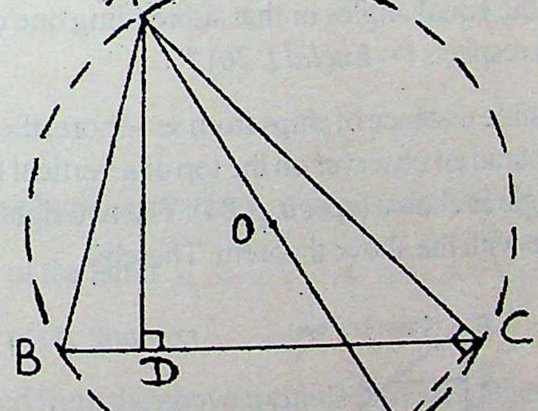

Figure 12

The result AB.AC=AD.AE easily follwos from the similar triangles ACE and ADB.

PROOF OF THE (SO-CALLED) PYTHAGORAS THEOREM

Gaṇesa Davjña's Buddhivilāsinī (c. 1545 A.D.), a commentary on the *LV*, attempts to furnish a proof [24] purely geometrical based on the property of similar triangles for Bhaskara II's geometrico-algebraic rationale of the above theorem.

SHADOW PROBLEMS

In ancient Hindu and Jaina texts, we generally come across two types of the shadow problems, one due to single gnomon and the other due to two equal gnomons (each based on the property of similar triangles). The latter appears in the *AB*, the *BrSS*, the *LV*, the *NP* and the *GK*, and also in the various commentaries on the *AB* by *Bhāskara I*, Suryadeva *Yajvān* (b. 1191 A.D.), *Parameśvara* (fl. 1380-1460 A.D.), Yallaya (ca. 1470 A.D.) and Raghunatha *Rāja*, whereas the former recurs in the various texts, viz. the *AB*, the *BrSS*, the *GSS*, *Sripati's Siddhāntaśekhara* (c. 1039 A.D.), the *LV*, the *NP* and the *GK*. For details, refer to Mishra and Singh [25].

Here is the Greek point of view :

Thales' (ca. 624-547 B.C.) geometrical theorem (without description) [26]:

"If two triangles have two angles equal to two angles respectively and one side to one side (namely that adjoining the equal angles or that subtending one of the equal angles, the triangles are equal in all respects (= *Euclid I. 26*)."

To measure inaccessible distance of ships from sea-shore, the one of the possibilities that Thales used was to stand an observer on the top of a vertical tower and constructing a small triangle in a triangle as shown (*op. cit.* p. 84). The two right triangles ABC & CDE are similar in accordance with the above theorem. Thereby

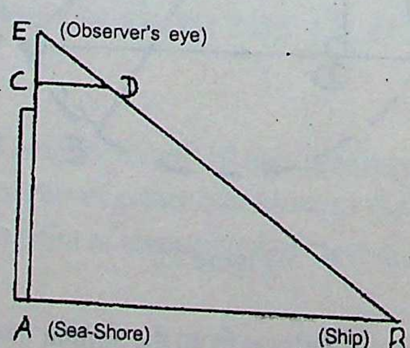

Figure 13

$$AB = \frac{AE.CD}{CE}$$

for if AE, CD & CE are measurable.

Now we come back to Indian case.

Shadow problems due to single gnomon

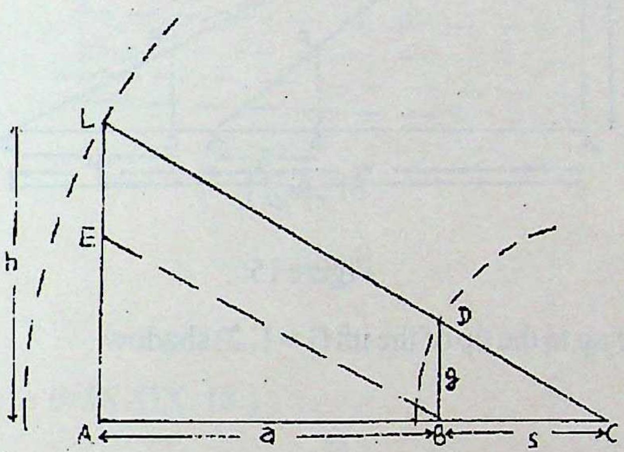

Figure 14

The Shadow-length

2) $s = \dfrac{ag}{h - g}$

wherein a = distance between the root of lamp-post and that of the gnomon

h = height of the lamp

g = height of the gnomon

[a and h can be evaluated from the above formula provided other parameters are known.]

The rationale follows from the similar triangles AEB and BDC.

Alternative explanation due to Nilkantha [27]: Source of light L is located on a circle concentric with the own circle (of gnomon and its shadow), C, the tip of the shadow, acts as centre and CD (D- the tip of the gnomon) as radius. Then LA, the vertical through L, is the sine-chord (*bhuja*) and AC (C- the foot of L) is the cosine-chord (*koṭi*).

Shadow problems due to two equal gnomons

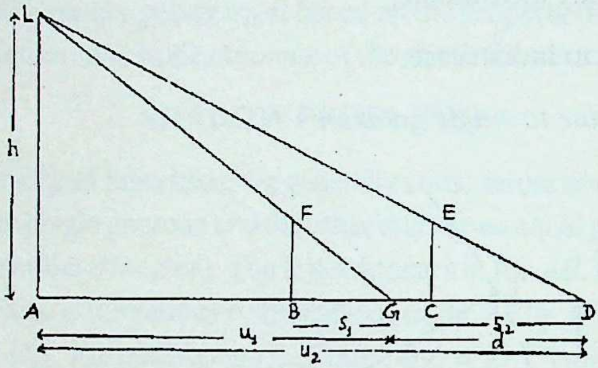

Figure 15

Upright corresponding to the tip of the ith (I = 1, 2) shadow

3) $u_i = ds_i / \Delta s$.

Height of the lamp-post

4) $h = g\,(u_i/s_i)$

$\Rightarrow h = g\,(d/\Delta s)$; $\Delta s = s_2 - s_1$, $\Delta u = u_2 - u_1 = d$.

The results are easily derivable using the properties of similar triangles. However *Parameśvara*, a commentator of the *Āryabhaṭīya,* bases his ratioale on the rule of three. Quite interesting problems and their excercises occur almost in all the above texts. However, we cite here only some enlightening (special) problems, other than the normal, from the *BrSS, LV, GK and GSS.*

I. Solutions to water-reflection problmes [28]

(a) *To find height and distance of the object by observing its reflection from a distance*

"The distance between the house and the man is divided by the sum of the heights of the house and the man's eyes and multiplied by the height of the eyes. The tip of the image of the house will be seen when the reflecting water is at a distance equal to the above product" [*BrSS* XIX.17].

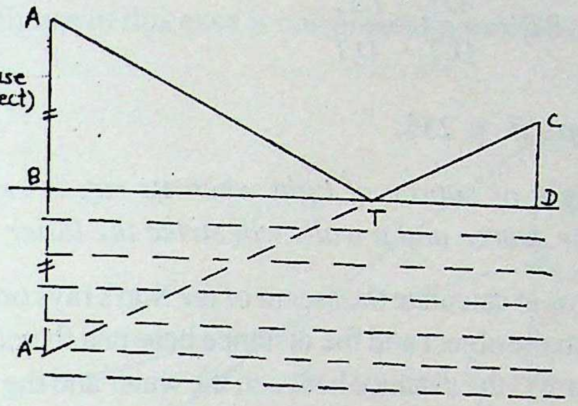

Figure 16

$$DT = \frac{BD.CD}{AB+CD}.$$

$$AB = \frac{BT.CD}{DT}$$ [due to *BrSS* XIX. 18.].

For rationale, refer to [*op. cit.* p. 254].

(b) *To find height and distance of the object by observing reflection from two different distances*

"The distance between the first and second positions of the water divided by the difference between the distances of the man from the water, when multiplied by the height of the eyes, is the height, and the same, when multiplied by the distance between the water and the man, is the distance between the water and the house" [*BrSS* XIX. 19].

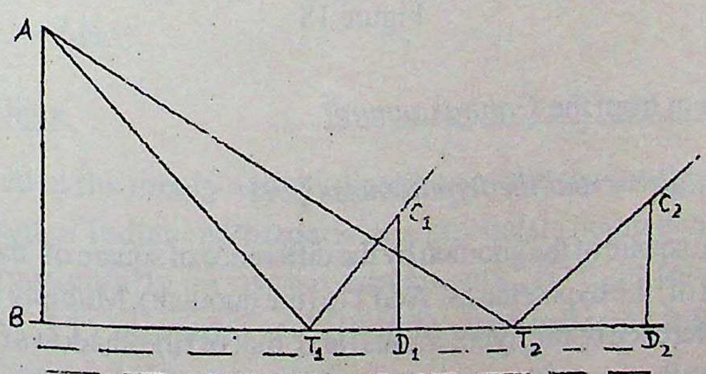

Figure 17

$$AB = \frac{C_1D_1.T_1T_2}{D_2T_2 - D_1T_1}, \quad BT_1 = \frac{DT_1 - T_1T_2}{D_2T_2 - D_1T_1}.$$

Consult for proof, *op. cit.*, p. 255.

(C) *To find the height or source of light when its ray is reflected from a water-surface (between the source and a wall) will strike the latter*

"He... who knows how to calculate the ascent of the Sun's rays on a wall from the known ratio of the shadow to the object and the distance between th water and the wall" [*BrSS* XIX. 8]. (In such a case) " the distance between the water and the wall divided by ratio of the shadow to the object is the height of ascent" [*BrSS* XIX.20].

$AB = \dfrac{BT}{DT/CD}$. See [*op. cit.*, p. 256] for its rationale.

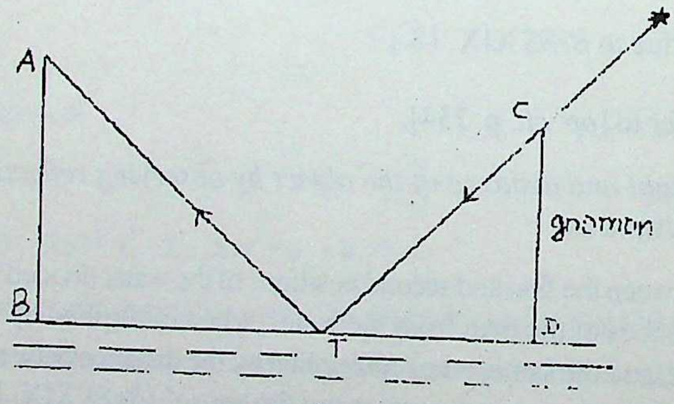

Figure 18

II. <u>Shadow problem from the *Gaṇitakaumudī*</u>

Lengths of the shadows and the hypotenuses [29]

"Divide 4 times the square of the gnomon by the difference of square of 'the difference of the shadow's and (that of) the hypotenuses'. Add 1 to (the quotient). Multiply the square-root of the sum by the difference of the hypotenuses (and, that of) the shadows (separately). (The products) happen to the sum of the shadows (and that of) the hypotenuses, respectively. The shadow and the hypotenuses should be obtained from them by *saṁkramaṇa*. The heights of the lights should be obtained by the method stated earlier" [*GK* 17-18].

[This problem seems to be an extension of the *LV* 232 which restricts to find shadow-lengths only. And the figure in this case is confined to gnomon, shadow-lengths and hypotenuses.]

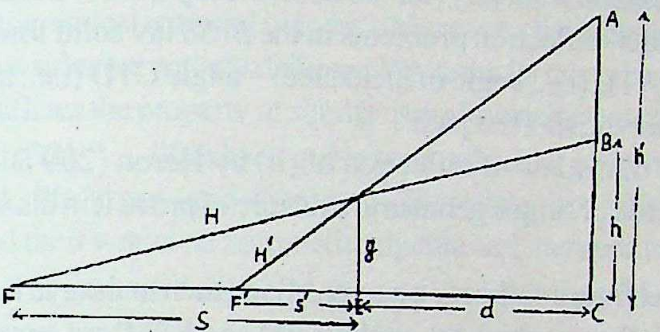

Figure 19

Let s - s' = a, H - H' = b, then according to the rule

$$s + s' = b\sqrt{\frac{4g}{a^2 - b^2} + 1} \text{ and } H + H' = a\sqrt{\frac{4g}{a^2 - b^2} + 1}.$$

Then by *saṁkramaṇa*

$$s \text{ or } s' = \frac{1}{2}\left[\pm a + b\sqrt{\frac{4g^2}{a^2 - b^2} + 1}\right], \text{ and}$$

$$H \text{ or } H' = \frac{1}{2}\left[\pm b + a\sqrt{\frac{4g^2}{a^2 - b^2} + 1}\right].$$

$\therefore$ h = dg/s and h' = dg/s'.

III. The nature of light

The light ray used in the theory and applications of the shadow problems certainly reflect the wisdom of Indian mathematicians/physicists being convergent with the nature of light particularly on its two aspects, viz., (i) light travels in a straight line (fifth century A.D. onwards), and (ii) light when reflected follows the shortest path (seventh century A.D. onwards).

Explanation

I am rather inconclusive whether the *Vedic* people were aware of the fact (i), but certainly after the dark age, in the first written treatise on mathematics of elder *Āryabhaṭa*, the shadow problems exhibit it silently (for instance, the ray LDC is a straight line, Fig. 12). Again the three water-reflection problems in the *BrSS* lay solid foundation to (ii), for instance, by angle ATB (i.e., angle of incidence) = angle CTD (i.e., angle of reflection) (Fig. 16) is meant for the shortest path.

Below is a proof[30] of the law of reflection of (ii) by Heron (200 B.C.) of Alexandria (Greek) who made use of simple geometric structure of prove it in his work on *Catoptrics* (i.e., reflection):

" If a light is to travel from a source to a mirror M' and then to the eye E of an observer, the shortest path SPE is that in which the angles SPM and EPM' are equal" [cf. 30].

Let the other path SP'E be feasibly short. Let SQS' $\perp$ MM' such that
SQ = S' Q. Join S' to P and P'. The path-lenghts
SPE = S' PE & SP'E = S' P' E.

Of the two, S' PE is a stright line (because angle M' PE = angle M' PS) and is, therefore, shorter than the bent line S' P' E. Hence path SPE < path SP' E.

The knowledge of (ii) is even credited to Euclid and Aristotle and sometime also to Plato (*op. cit.*, p. 194).

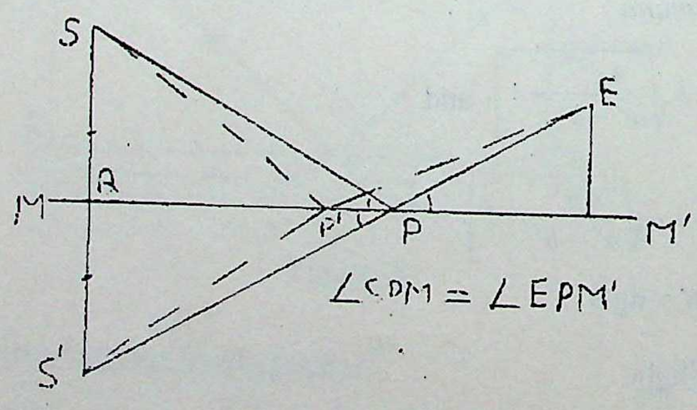

Figure 20

CONCLUDING REMARKS

Various studies reveal that Buddhism was the main vehicle for transmission of Indian culture to China. During the intercourse, a large number of Indian works were translated into Chinese language between first and early eighth century A.D.

The parallel problems on broken bamboo, lotus among others in India throw some light on the possible Chinese transmission to India. However, according to Gupta [31]," ..., while things were documented in Chinese sources, there is no familiar positive literary or other documentary evidence known from Indian sources which specify clearly the arrival of any Chinese mathematical material in India". Moreover, the techniques employed for solution from the two sides are entirely different. While the former is based on Pythagorean property, the later utilizes the property of similar triangles arising out of a chord in a circle. According to Maiti [32], " ... *Bhāskara* quotes one *gāthā* in *prākrta*, which states the rule for the chord, the arrow and diameter. Therefore, his reference to previous mathematicians and their works on arithmetic, algebra and mensuration during or before *Āryabhaṭa* leads us to assume that *vaṁśabhaṅga* problem may have been devised by either *Maskarī* or *Pūraṇa* or Mudgala or Putana. And if *Maskarī*, the mathematician be the Makkhali of the Buddhists or Mamkhali of the Jainas or *Maskarī* of Panini and *Pūraṇa* Kassapa or *Pūraṇa* of the *Mahābhārata*, then *śyenamūsika* problems along with *vaṁśabhaṅga* may have their origin in the Buddha's time, at least. This may not be an impossible probability because certain numeral terms are found to occur in *Maskarī*'s creed and these may have been made use of, in all probability, in the pursuit of his system of principle. Further, measurements with bamboo were extensively known during *Brāhamaṇa* and *Śulba* periods. *Āpastamba* had made use of a bamboo-rod to construct a square having a given side. In the time of *Arthaśāstra* (400 B. C.) the length measure *daṇḍa* is found to have been in common use. An interesting story of measuring a Buddha statue with a bamboo pole had been referred to by the author of the *She-Kia-Fang-Che:* "In former times there was a man who with a sixteen feet pole tried to measure Buddha". These facts, although meagre, also allude to the probability of setting up the problem by Indian Mathematician themselves. The Jaina *Syādvāda* (Seven Fold Prediction) in which lies the seed of the theory of the probability also suggests India's mathematical attainment to a comparatively high degree in 400 B.C. ".

Noticiable that, as pointed out by Van der Waerden[33] (see Bag [11]), Babylonian text BM 34568 (1900-1600 B.C.) - according to his classification - gives 13 types of problems involving $x^2 + y^2 = z^2$, and that the bamboo problem comes under type VI : Given $x = a$, and $y + z = b$, to find x and z. More interestingly the four problems of the above Babylonian text are found to be indentical with the Chinese type 1, 2, 3, and 6 [26]. After a detailed comparative study Van der Waerden "conjectures a common origin of mathematics" which he called pre-Babylonian mathematics (cf. Bag, ref. 33).

The remaining problems other than the above two enjoy the full credit of being Indian

contribution and free from any foreign influence.

The various geometrical problems concerning light confirm the Indian perception of the two basic laws of it certainly after the work of Brahmagupta. In this context the remark by T. A. Sarasvati ([28], p. 256)' ... the possibility of Brahmagupta having investigated the laws of reflection' is worth noting.

SUPPLIMENT

In the *Śulbasūtras*, many foundational aspects of the knowledge of similar figures are practically observed due to altar construction. The one -" the corresponding sides and lines of similar figures are proportionate " - for example, has an interesting application in the construction of the bricks to be used in the wing of the *vakrapasa-śyenacit* (a falcon shaped altar with curved wings.) *Āpastamba* says, "The traverse side is $1/7^{th}$ of the side of the side of the wing and the lateral side is 1/4 of a *puruṣa*. Its frame should be expanded diagonal-wise. The planks should be inclined by $1/7^{th}$ of the *pakṣanamanī* -the shape or gradient of the wing" [Sarasvati, [27], pp. 48-49].

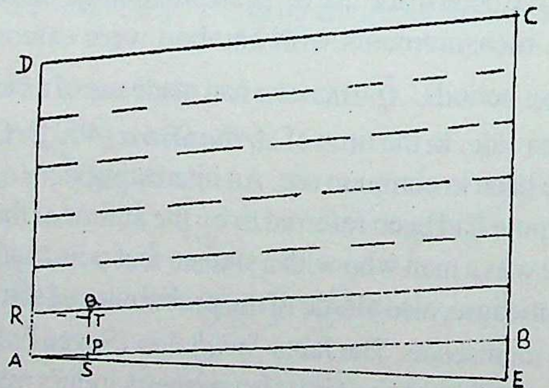

Figure 21

Here *pakṣanamanī* means for the vertical height of the tip of the inclined wing from the horizontal. ABCD is half of the wing as shown. CD is produced to meet the horizontal through A in E. BE is the *pakṣanamanī*, here. If APQR is a brick, QP is extended to meet the horizontal through A is S. The triangle APS is similar to the triangle ABE. Therefore

$$\frac{PS}{BE} = \frac{AP}{AB} = \frac{1}{7} \; or \; PS = \frac{1}{7} BE.$$

Application of the rule (theorem of square on half chord) to find the radius of curvature *(vakratārdhyavyāsa)* of spherical surface

Corresponding to 'a', half of the chord (*jivādharm*), and 'h', arrow (*śara*), the radius of the circle (*vṛta*) will be governed by the formula

$$R = \frac{a^2}{2h} + \frac{h}{2}.$$

To find the radius of curvature of a spherical surface, we need to compute, 'a' and 'h' with perfection, for which in modern days, an instrument called *spherometer* is used. A spherometer basically consists of three equal equilateral legs and a central leg capable of moving vertically with the help of metallic circular plate. Very firstly the tip of the central leg is brought in contact with the plane glass plate and distance between the tips of central and outer legs, 'a' is measured. It is then placed gently on the spherical surface rotating slowly the screw so as to move central leg upward till three outer legs are just in touch with the surface, the net upward movement 'h' is measured from the gratuated scale and the marked metallic disc. The final setting of the spherometer and the spherical surface is as shown in the Figure. [N. G. Dongre, *Physics in Ancient India*, New Age International Publishers, New Delhi, 1994, 48-49].

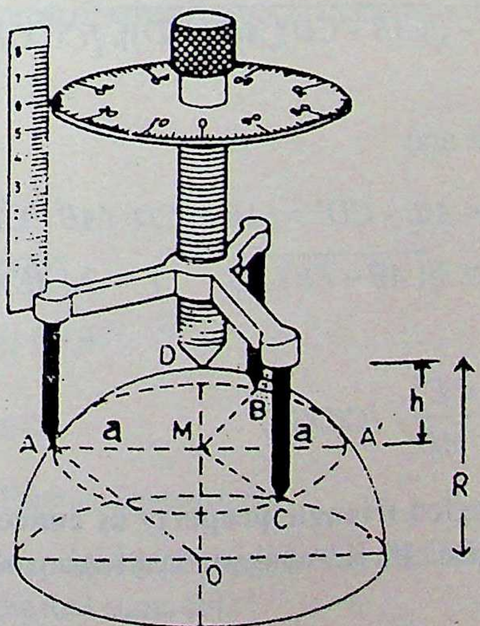

Figure 22

Algebraic proof of eq. (1) (as in the *LV*) [K.S. Patwardhan, S. A. Naimpally and S. L. Singh, *Līlāvatī of Bhāskarācārya*, Motilal Banarsidass Publishers, Delhi, 2001, pp. 146- 147]

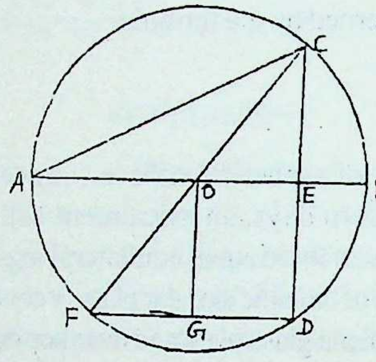

Figure 23

$CO = OB.$

$\Rightarrow CO - OE = OB - OE = EB.$

$\therefore\ AE = AO + OE = CO + OE = \dfrac{CO^2 - OE^2}{CO - OE} = \dfrac{CE^2}{EB}.$

$EB = OB - OE = \dfrac{1}{2}[AB - \sqrt{(AB + CD)(AB - CD)}].$ *[CCXIII, Ist half]*

[In it, $OE = GD = \dfrac{1}{2}\ FD$, and

$$FD^2 = CF^2 - CD^2 = AB^2 - CD^2 = (AB + CD)\ (AB - CD).]$$

$CD = 2\ CE = 2AE.\ EB = 2\sqrt{(AB - EB).EB}\qquad [= 2\sqrt{AE.(AB - AE)}.$

[CCXIII, 2nd half]

$AB = AE + EB = AE + \dfrac{CE^2}{AE}.$ *[CCXIV]*

Application of right-angled triangle property as conceived in old Babylonia/ Summeria of about 1900 B.C. [S. K. Adhikari, Babylonian mathematics, *Indian J. Hist. Sc.* 1998, 33, pp. 17-18].

From a Babylonian cuneiform text (the text name is not mentioned by Adhikari) :

To find the length of the chord of a circle when diameter and depth of the sector is respectively (d=) 20 and (h =) 2.

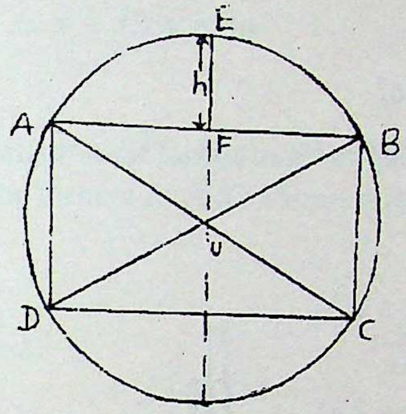

Figure 24

$AB^2 = AC^2 - BC^2\ [= d^2 - [d - 2h)^2 = 4\,(d - h)h]$ is the same as eq. (1).

$AC = d = 20,\ BC = d - 2h\ \ 20 - 4 = 16.$

$\therefore AB = \sqrt{AC^2 - BC^2} = 12.$

The method of solution as prescribed can be put in modern form as :

$2 \times 2 = 4$

$20 - 4 = 16$

$20 \times 20 = 6\,(40)\ [= 400]$

$16 \times 16 = 4\,(16)\ [= 256]$

$6\,(40) - 4\,(16) = 2\,(24)\ [= 48]$

$\sqrt{2(24)} = 12$ is the chord.

<u>From a Summerian tablet (Thureau Dangin text)</u>

"A straight rod 30 in. stands upright then it slips so that the upper end slides 6 in. down a vertical wall. How far has the foot come out?"

In Babylonian way the solution can be put as:

Patu 30

$30 - 6 = 24$

$30 \times 30 = 15\ [= 900]$

$24 \times 24 = 9\ (36)\ [= 576]$

$15 - 9\ (36) = 5\ (4)\ [=324]$

$\sqrt{5(24)} = 18$

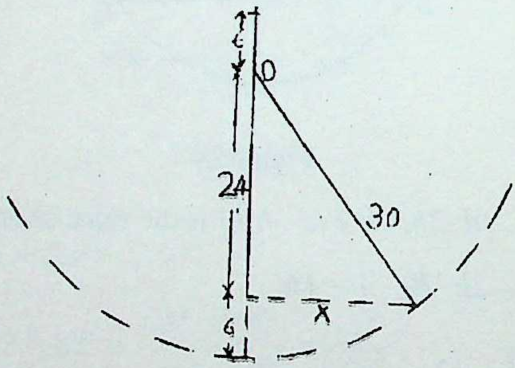

Figure 25

In the Figure,

$x = \sqrt{(d-h)h} = \sqrt{(60-6)6} = \sqrt{324} = 18.$

Note. A number within [ ] means its modern equivalent.

BRIEF SOLUTIONS TO EXAMPLES ON THEOREM OF SQUARE ON HALF-CHORD

Broken bamboo problem

Eq. *(1)* $\Rightarrow EB = CE^2/AE = 36/18 = 2.$

$x + y = 18,\ y - x = 2 \Rightarrow x = 8,\ y = 10.$

Parallel Chines problem

$y^2 = z^2 - 32 = (y - 10)^2 - 9 \Rightarrow y = 91/20 = 4\frac{11}{20}.$

Lotus problem

Eq. *(1)* $\Rightarrow AE = CE^2 / EB = 576/8 = 72.$

$x + y = 72, y - x = 8 \Rightarrow x = 32, y = 40.$

Ṭabarī's problem

As reported, his solution in algorithmic form is based on the property of intersecting chords in a circle and is given in the form of Euclid's *Elements Book III (Theorem 34),* without any explanation.

$9 \times 9 = 81$	$[CE \times CE = CE^2]$
$81 : 3 = 27$	$[CE^2/EB = AE]$
$27+3 = 30$	$[AE + EB = AB]$
$30 : 2 = 15$	$[AB/2 = OB]$

***Al-Kāshī*'s problem**

According to *al-Kāshī* (following *Ṭabarī*) [cf. 15]

"Square of CE divided by EB gives EA : 5 x 5 : 3 = 25 /3. Adding EB gives BA = 34/3. This is the diameter, and by halving it the radius is found, which equals the length of the spear : 34/3 : 2 = 17/3 = 5 + 2/3 cubits."

Al-Kāshī does not hesitate to present algebraic methodology for its solution which is as follows [cf. *op. cit.*] :

$EO = x, x^2 + 25 = CO^2 = OB^2 = (x + 3)^2$

$x^2 + 25 = x^2 + 6x + 9, 6x = 16, x = 8/3$

$8/3 + 3 = 17/3 = 5 + 2/3.$

Parallel Chinese problem

$x = (z^2 - e^2) / 2e = (25 - 1) / 2 = 12.$

Hawk and rat problem

Eq. (1) $\Rightarrow EB = CE^2 / AE = 324/ 81 = 4.$

$CO = 42\frac{1}{2}, \; OE = 38\frac{1}{2}.$

Crane and fish problem

$EB = FE^2/AE = FE^2/(CD+DE) = 36/18 = 2.$

$OF = AO = AB/2 = (AE+EB)/2 = (18+2)/2 = 10.$

Bamboo problem

h = 6 feet follows from the similar triangles BCD & BLM and ABD & LMD.

$h/15 = x/(x+y)$ and $h/10 = y/(x+y) \Rightarrow y = \frac{3}{2}x.$

SECTION B

INTRODUCTION

This part is written with the intention of John Stillwell that "... ancient mathematics is not dead or absolete- it stays alive because its problems continues to stimulate the creation of new concepts and techniques. The most fertile problems in mathematics are over 2000 years old and still not yielded up all their secrets" (p.1, reference [4]). Indian mathematics is proud of his genius Brahmagupta (b. 598 A.D.) for beautiful construction of a cyclic quadrilateral with perpendicular diagonals from two pythagoeran triplets. For details of further study in the field, refer to Mishra and Singh [3]. The intent of this part is to reveal the further facts utilizing the concepts of geometry and group theory when two adjacent sides of the prescribed quadrilateral corresponding to a vertex of circum-diameter are equal; the two other sides corresponding to the second vertex of the same diameter has to be different.

GEOMETRIC STUDY

Let (a_0, b_0, c_0) and (a, b, c) be two pythagorean triplets with $a_0 > b_0$, $a > b$. These triplets readily supply the sides of cyclic quadrilateral A'BC'D with perpendicular diagonals, according to Brahmagupta, as $C'D = a = a_0c$, $BC' = b = ac_0$, $A'B = ¢ = b_0c$ & $DA' = d = bc_0$, taken in order. The diagonals are :

$d_1 = AC = AL + LC = b_0b + a_0a.$

$d_2 = BD = BL + LE = ab_0 + a_0b.$

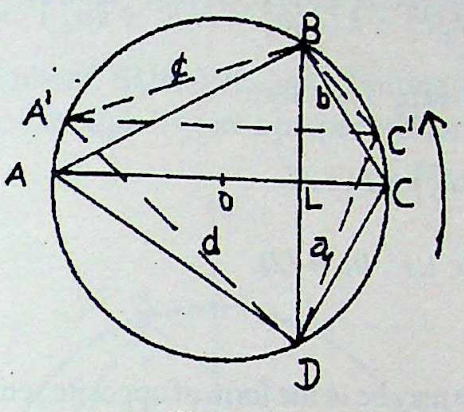

Figure 1a

This structure leads to the following theorem :

Theorem. In a cyclic quadrilateral ABCD with sides as above, the following statements are equivalent for $A' \underset{\rightarrow}{=} A,\ C' \underset{\rightarrow}{=} C$.

a) Angle at B (and angle at D) = a right angle (each).

b) $AB^2 + BC^2 = AC^2$ (and $AD^2 + DC^2 = AC^2$).

c) $BL = LD$, *i.e.*, $a/a_0 = b/b_0$.

d) $BL^2 = AL.LC$.

Proof. For (a) $\Rightarrow$ (b) & (b) $\Rightarrow$ (a), the proof is well-known and need no reproduction.

For (b) $\Rightarrow$ $(c), AB^2 + BC^2 = AC^2$, *i.e*,

$$b_0\left(a^2 + b^2\right) + a^2\left(a_0^2 + b_o^2\right) = \left(bb_0 + a_0a\right)^2.$$

$\Rightarrow ab_0 = a_0b$, *i.e.*, $BL = LD$.

For $(c) \Rightarrow (b)$, *let* $BL = LD$, *i.e*, $ab_0 = a_ob$ or $a/a_0 = b/b_0 = k$, say;

$\Rightarrow a = ka_0, b = kb_0$ & so $c = kc_0$; $¢ = d = kb_0c_0$ & $b = a = ka_0c_0$. Notice that ¢ < b since $b_0 < a_0$.

Now $AB^2 + BC^2 = k^2b_0^2c_0^2 + k^2a_0^2c_0^2 = (kb_0^2 + ka_0^2)^2 = AC^2$.

For (c) $\Rightarrow (d)$, $a = ka_0$, $b = kb_0$. Therefore

$$BL^2 = (ka_0b_0)^2 = (ka_0^2)(kb_0^2) = LC.AL.$$

This implies that $ab_0 = a_0b$, *i.e.*, $BL = LD$.
This completes the proof.

Remark. The above figure may be in the form of opposite sense.

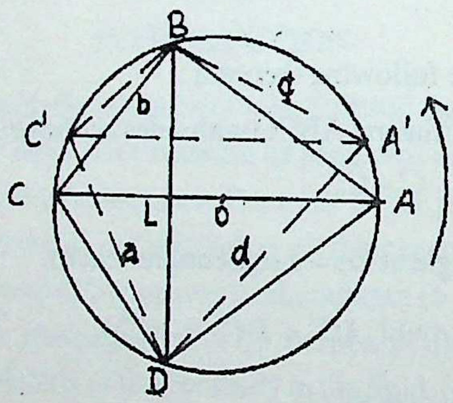

Figure 1b

Conclusion. In nut shell, we see that if $(a_0, b_0\ c_0)$ is a primitive trianlge with $a_0 > b_0$, and k any scalar, then it is always possible to construct a cyclic quadrilateral with two adjacent sides ka_0c_0 each through a vertex of the diagonal of the circle (i.e., diagonal of the quadrilateral) and other two sides kb_0c_0 each through other vertex of the same diagonal. The circum-diameter is given by kc_0^2, and the perpendicular diagonal by $2ka_0b_0$.

Geometric construction otherwise and deduction therefrom

Figure 1a or 1b with four points ABCD possesses reflexive symmetry about AC. That is, any study of half-plane (semi-circle) will automatically reveal the secrets of the quadrilateral under consideration. The study of semi-circle by drawing a perpendicular from a point on half circle onto the diameter has several references in theory and applications in ancient mathematical world of Arab, Babylonia/Summeria.

China, India and Greek. The study had been done so far using Pythagoras theorem and similarity concept.

Let AL bc a line segment. Extend AL to C. Letting O the midpoint of AC as centre draw a semi-circle. Through L erect a perpendicular BL on AC.

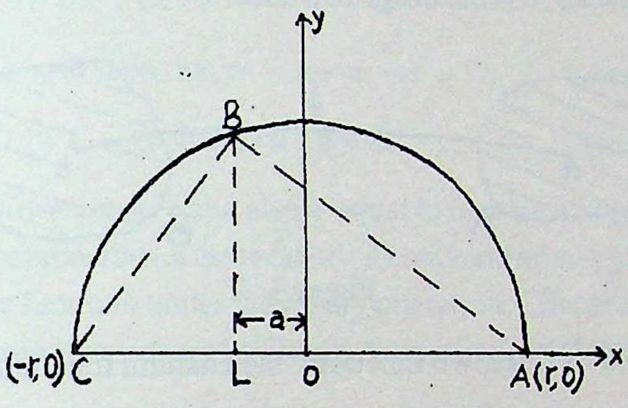

Figure 2

Equation of semi-circle through ABC

(1) $x^2 + y^2 = z^2$, $y > o$.

Equation of line BL parallel to x = o

(2) x = - a.

Using (2) in (1),

(3) $BL^2 = r^2 - a^2 = (r + a)(r - a) = AL.LC.$

In particular if $LC = 1$, then $BL = \sqrt{AL}.$ This implies that if $\alpha(= AL)$ is constructible then so is $\sqrt{\alpha}(= BL)$. Notice that " a real number α is said to be *constructible* if one can construct a line segment of length α in a finite number of steps from this segment of unit length by using a straightedge and a compass" [1, p. 354].

GROUP THEORETIC STUDY

In study of permutation groups, the defining terms of function, composite function, permutation and group are of utmost importance. For greater insight, these are explained briefly as hereunder. For the sake of completeness, we recall certain elementary definious.

Assume A and B are two non-empty sets.

A *function* or *mapping* ϕ from A into B (symbolically $\phi : A \rightarrow B$) is a rulc that assigns to each element a of A a unique element b of B. We say that ϕ maps a into b and is written as $\phi(a) = b$. Equivalently, b is the image of a under ϕ.

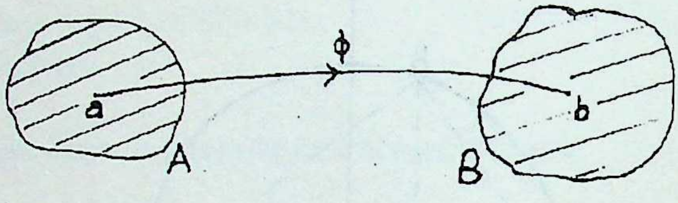

Figure 3

If $\phi : A \rightarrow B$ *and* $\phi : B \rightarrow C$ are two functions, then natural function mapping A into C consists of ϕ followed by φ. For $\phi(a)=b, \varphi(b) = c$; $(\varphi\ \phi)(a) = \varphi(\phi(a) = \varphi(b) = c$. $\varphi\ \phi$ is a *composite function.*

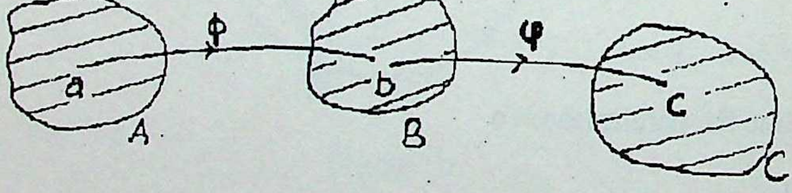

Figure 4

We call a function from A into B *one to one* (or *injective)* if each element of B has atmost one element of a mapped into it, and *onto* (or *surjective)* if each element of B has atleast one element of A mapped into it.

A *permutation* of a set A into itself is a *bijective* map, i.e. one to one function from A onto A. In this case, each element in the image appears once and only once. For instance, {2, 4, 5, 1, 3} is one of the permultations of the set {1, 2, 3, 4, 5} out of 5!, but not {3, 2, 5, 4, 3}.

A *binary* operaton on a non-empty set S (or a *composition* in S) is basically a function $\phi: S \times S \rightarrow S$. The image $\phi(a, b) \in S$ for a, b $\in$ S is usually denoted convincingly by ab or a.b (multiplicative notation), a + b (additive notation), a*b etc. A binary operation on S is

equivalent to the statement: If a, b $\in$ S then ab $\in$ S (closure axiom).

A *group G* is an ordered pair (G, .) consisting of a non-empty set G together with a binary operation on G satisfying the properties:

i) $(ab)c = a(bc)\ \forall a,b,c \in G$; (associativite law)

ii) $\exists$ an element e $\in$ G such that ea = ae = a $\forall$ a $\in$ G; (existence of identity)

iii) for each a $\in$ G, $\exists$ a' $\in$ G such that a'a = aa' = e. (existence of inverses)

Notice that the *identity element* e and also a' equal to *inverse* of a (denoted by a^{-1}) are unique. Eventually, the properties of associativity, identity and inverse altogether determine a group of one to one function under the binary operation. [The abstract definition of group is due to Leopold Kronecker (1870) and Arthur Claley (1878).

A group G is said to be *commutative* iff its binary operation is commutative, i.e., ab = ba for all a, b $\in$ G. This group is sometimes called *abelian* group (named in honour of N. H. Abel (1802-1829) of Norway who did remarkable work in this field).

Theorem. Let A be a non-empty set and S_A the collection of all permutations of A. Then S_A is a group under permutation multiplication.

Mirror. In Figure 5, line *m* acts as a mirror (or a line of reflection), i.e., every point A of the plane on m acts as its own image, i.e., mapped to itself $(\sigma(A) = A)$ under transformation σ. Such a point is the fixed point of σ. And the image of a point P off m to P', σ (P) on the opposite side of m. Eventually, m serves as a right bisector of the line PP'. Also distance and angle measure are preserved (i.e., [R, S] = [R', S'] and angle RST = angle R'S'T' in reversing sense). In other words, every line m deterines an injective map of the plane.

Mirror axiom. For every line m there exists a collineation that preserves distance and angle measure, fixes m pointwise, and interchanges the half planes of *m*.

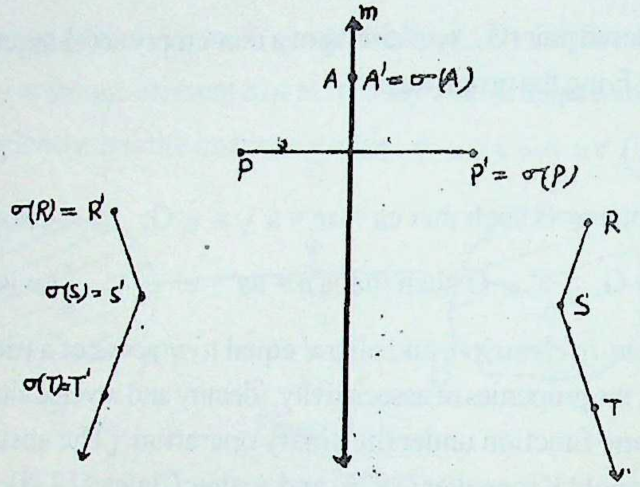

Figure 5

Assume set A = {α, α, β, β}, taken in order of 1 to 4 to avoid any mistake in further opcrations, denotes the side-lengths of a cyclic quadrilatcral, where $\alpha = k\beta_o\gamma_0$ & $\beta = k\alpha_o\gamma_o$; $(\alpha_0\beta_{0,}\gamma_o)$ is a Pythagorean triplet $(\alpha_0 < \beta_0)$ and k a scalar. For a systematic study of group of permutations of A, we need to symbolize ρ_i for permutations corresponding to sides of Figure 6 and μ_j for permutations corresponding to the mirror image of this figure. The eight elements of S_8, the collection of all permutations of A, are:

$$I = \rho_o(1234) = \begin{bmatrix} a & a & b & b \\ a & a & b & b \end{bmatrix} = \mu_2(2143)$$

$$\rho_1(2341) = \begin{bmatrix} a & a & b & b \\ a & b & b & a \end{bmatrix} = \mu_1(1432)$$

$$\rho_2(3412) = \begin{bmatrix} a & a & b & b \\ b & b & a & a \end{bmatrix} = \mu_4(4321)$$

$$\rho_3(4123) = \begin{bmatrix} a & a & b & b \\ b & a & a & b \end{bmatrix} = \mu_3(3214)$$

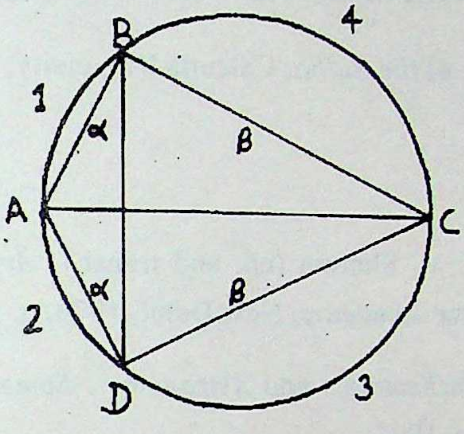

Figure 6

There is one to one correspondence among $\rho_i s$ and $\mu_j s$ in a manner

$$\rho_1 \leftrightarrow \mu_1;\ \rho_2 \leftrightarrow \mu_4;\ \rho_3 \leftrightarrow \mu_3; \qquad I = \rho_4 \leftrightarrow \mu_2$$

and is due to the fact of symmetry about circum-diameter AC. The choice of *i* and *j* in ρ and μ are as per convenience.

The credit for definition of group of permutations and 2- row notation goes to the French mathematician Augustin-Louis Cauchy (1789-1857).

Notice that, for instance,

$$\rho_1\mu_3 = \mu_{2.}\mu_3\rho_1 = \mu_4,\ \textbf{\textit{i.e.}},\ \rho_1\mu_3 \neq \mu_3\rho'$$

$$\left[by\ \rho_\rho\rho_{q}(A_i) = \rho_\rho\left(\rho_q(A_i)\right)\right] = \rho_p(A_i) = A_{k,}\ 1 \le i, j \le 4].$$

Hence S_8 (octic group) is non-abelian.

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A COMMON FIXED POINT THEOREM IN FUZZY METRIC SPACES

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ABSTRACT

Using notions of compatibility and R-weak commutativity, we prove a common fixed point theorem for six mappings in fuzzy metric spaces. Our result generalizes and fuzzifies some of the recent results in metric and fuzzy metric spaces. A related example is also furnished.

AMS subject Classification (2000) : 47H10, 54H25, 54E70.

Keywords : Fuzzy metric space, compatibility, R-weakly commuting, common fixed point.

INTRODUCTION

The theory of fuzzy sets was introduced by L. Zadeh [25] in 1965. Kramosil and Michalek [18] introduced the fuzzy metric space by generalizing the concept of probabilistic metrc space to fuzzy situation. Deng [5], Erceg [6] and Kaleva and Seikkala [17] have defined fuzzy metric spaces in different ways. George and Veeramani [8] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [18].

Grabiec [9] proved the contraction principle in the setting of fuzzy metric spaces introduced in [18]. For fixed point theorems in fuzzy metric spaces some of the interesting references are [1-4], [7], [9-11], [13], [14], [19], [21,22,24] etc.

Jungck [15, 16] gave the more generalized concept of compatibility than commutativity and weak commutativity in metric spaces. Mishra, Sharma and Singh [19] obtained common fixed point theorems for compatible maps on fuzzy metric spaces. Recently, Vasuki [24] introduced the concept of R-weak commutativity of

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mappings in fuzzy metric space. For an excellent comparision of weaker forms of commuting maps, one may refere to Singh and Tomar [23].

In the present paper, using notions of compatibility and R-weak commutativity, we establish a common fixed point theorem for six mappings in fuzzy metric spaces in the sense of Kramosil and Michalek. Our result unifies, generalizes and fuzzifies some recent results due to Sharma [21], Singh and Chauhan [22] and Imdad and Khan [12] with less restrictive conditions on mappings.

PRELIMINARIES

Definition 1 [20]. A binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if $([0, 1], *)$ is a topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Examples of t-norm are $a * b = ab$ and $a * b = \min \{a, b\}$.

Definition 2 [18]. The triplet $(X, M, *)$ is a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, M is a fuzzy set in $X^2 \times [0,1]$ satisfying the following conditions:

(2.1) $M(x, y, 0) = 0$,

(2.2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,

(2.3) $M(x, y, t) = M(y, x, t) \neq 0$ for $t \neq 0$,

(2.4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$,

(2.5) $M(x, y, .) : [0, 1] \rightarrow [0, 1]$ is left continuous for all $x, y, z \in X$ and $s, t > 0$,

(2.6) $\lim_{t \to \infty} M(x, y, t) = 1$ for all x, y in X.

Lemma 1 [9]. For all $x, y \in X$, $M(x, y, *)$ is nondecreasing.

Definition 3 [9]. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is a Cauchy sequence if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that

$$M(x_n, x_m, t) > 1 - \varepsilon \text{ for all } n, m \geq n_0.$$

Definition 4 [9]. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to converge to x if and only if for each $\varepsilon > 0$, $t \geq 0$, there exists $n_0 \in N$ such that

$$M(x_n, x, t) > 1 - \varepsilon \quad \text{for all } n \geq n_0.$$

Definition 5 [22]. Self mappings F and G of a fuzzy metric space $(X, M, *)$ are said to be compatible if and only if $M(FGx_n, GFx_n, t)\ 1 \rightarrow$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Fx_n, Gx_n \rightarrow y$ for some y in X.

Definition 6 [24]. The mappings F and G of a fuzzy metric space $(X, M, *)$ into itself are R-weakly commuting provided there exists some positive real number R such that

$$M(FGx, GFx, tR) \geq M(Fx, Gx, t) \text{ for all x in X.}$$

Obviously, weak commutativity implies R-weak commutativity. However, R-weak commutativity implies weak commutativity only when $R \leq 1$.

Lemma 2 [3] . Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. If there exists a number $k \in (0, 1)$ such that

$$M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t) \text{ for all } t > 0 \text{ and } n = 1, 2, \ldots\ldots,$$

then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 3 [19]. If for all $x, y \in X$, $t > 0$ and for a number $k \in (0, 1)$,

$$M(x, y, kt) \geq M(x, y, t) \text{ then } x = y.$$

MAIN RESULTS

Theorem 1. Let A, B, S, T, I and J be self mappings of a complete fuzzy metric space $(X, M, *)$ with continuous t-norm defined by $a * b = \min\{a, b\}$, $a, b \in [0, 1]$, satisfying the following conditions :

(1.1) $AB(X) \subset J(X)$, $ST(X) \subset I(X)$,

(1.2) if either (a) $\{AB, I\}$ are compatible, I or AB is continuous and (ST, J) are R-weakly commuting, or

(a') {ST, J} are compatible, J or ST is continuous and (AB, I) are R-weakly commuting,

(1.3) there exists a number $k \in (0, 1)$ such that

$$M(ABx, STy, kt) \geq \min \{M(ABx, Jy, 2t), M(STy, Ix, t), M(ABx, Ix, t), M(STy, Jy, t), M(Ix, Jy, t)\} \text{ for all } x, y \text{ in } X \text{ and } t > 0.$$

(1.4) $\lim M(x, y, t) \to 1$ as $t \to \infty$, for all x, y in X; then AB, ST, I and J have a unique common fixed point. Furthermore, if the pairs (A, B), (A, I), (B, I), (S, T), (S, J) and (T, J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $AB(X) \subset J(X)$, we can find a point x_1 in X such that $ABx_0 = Jx_1$. Also since $ST(X) \subset I(X)$ we can choose a point x_2 with $STx_1 = Ix_2$. Using this argument repeatedly one can construct a sequence $\{y_n\}$ such that

$$y_{2n} = ABx_{2n} = Jx_{2n+1}, \; y_{2n+1} = STx_{2n+1} = Ix_{2n+2}$$

for n = 0, 1, 2,...

From (1.3)

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) &= M(STx_{2n+1}, ABx_{2n+2}, kt) \\ &\geq \min \{M(ABx_{2n+2}, Jx_{2n+1}, 2t), \; M(STx_{2n+1}, Ix_{2n+2}, t), \\ &\qquad M(ABx_{2n+2}, Ix_{2n+2}, t), M(STx_{2n+1}, Jx_{2n+1}, t), \\ &\qquad M(Ix_{2n+2}, Jx_{2n+1}, t)\} \\ &= \min \{M(y_{2n+2}, y_{2n}, 2t), \; M(y_{2n+1}, y_{2n+1}, t), \\ &\qquad M(y_{2n+2}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+1}, y_{2n}, t)\} \\ &= \min \{M(y_{2n}, y_{2n+1}, t), \; M(y_{2n}, y_{2n+2}, 2t), 1\}, \end{aligned}$$

$$\geq \min \{M(y_{2n}, y_{2n+1}, t),\ M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n}, y_{2n+1}, t)\}.$$

In general

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t).$$

By Lemma 2, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, $\{y_n\}$ converges to some point z in X. Thus $\{ABx_{2n+1}\}$, $\{Jx_{2n+1}\}$, $\{STx_{2n+1}\}$ and $\{Ix_{2n+2}\}$ also converge to the point z.

Case I : Let us now assume that I is continuous. Then the sequences $\{I^2x_{2n}\}$ and $\{IABx_{2n}\}$ converge to Iz. Thus for $t > 0$, $\lambda \in (0,1)$, there exists an integer $n_o \in N$ such that

$$M(IABx_{2n}, Iz, t/2) > 1 - \lambda\ ,\ M(I^2x_{2n}, Iz, t/2) > 1 - \lambda \text{ for all } n \geq n_0. \quad (1.5)$$

As $\{AB, I\}$ are compatible, using (1.5) we get,

$$M(ABIx_{2n}, Iz, t) \geq \min \{M(ABx_{2n}, IABx_{2n}, t/2), M(IABx_{2n}, Iz, t/2)\}$$

$$> 1 - \lambda \text{ for all } n \geq n_0.$$

Hence $ABIx_{2n} \to Iz$. (1.6)

Now using (1.3),

$$M(ABIx_{2n}, STx_{2n+1}, kt) \geq \min \{M(ABIx_{2n}, Jx_{2n+1}, 2t), M(STx_{2n+1}, I^2x_{2n}, t),$$

$$M(ABIx_{2n}, I^2x_{2n}, t), M(STx_{2n+1}, Jx_{2n+1}, t), M(I^2x_{2n}, Jx_{2n+1}, t)\}.$$

On letting $n \to \infty$ and using (1.6), we get

$$M(Iz, z, kt) \geq M(Iz, z, t).$$

So by Lemma 2, $Iz = z$. (1.7)

Again using (1.3),

$$M(ABz, STx_{2n+1}, kt) \geq \min\{M(ABz, Jx_{2n+1}, 2t), M(STx_{2n+1}, Iz, t), M(ABz, Iz, t), M(STx_{2n+1}, Jx_{2n+1}, t), M(Iz, Jx_{2n+1}, t)\}.$$

Letting $n \to \infty$ and using (1.7),

$$M(ABz, z, 2t) \geq \min\{M(ABz, z, 2t), M(z, Iz, t), M(ABz, Iz, t), M(z, z, t), M(Iz, z, t)\}$$
$$= M(ABz, z, t).$$

Therefore $M(ABz, z, kt) \geq M(ABz, z, t)$. By Lemma 3, this implies that

$$ABz = Iz = z. \tag{1.8}$$

Since $AB(X) \subset J(X)$, there always exists a point z' such that

$J(z') = AB(z) = z$ so that $STz = ST(Jz')$.

Now

$$M(z, STz', kt) = M(ABz, STz', kt)$$
$$\geq \min\{M(ABz, Jz', 2t), M(STz', Iz, t), M(ABz, Iz, t), M(STz', Jz', t), M(Iz, Jz', t)\}$$
$$\geq M(STz', z, t).$$

Hence by Lemma 3 $STz' = z = Jz'$, (1.9)

which shows that z' is the coincidence point of ST and J. Now using the R-weak commutativity of (ST, J) and from (1.9) we have,

$$M(STJz', JSTz', Rt) \geq M(STz', Jz', t) \to 1.$$

Therefore $ST(Jz') = J(STz')$.

Hence $STz = ST(Jz') = JSTz' = Jz$, (1.10)

which shows that z is also coincidence point of the pair (ST, J).

Using (1.3), (1.7), (1.8) and (1.10),

$$M(z, STz, kt) = M(ABz, STz, kt)$$
$$\geq \min \{M(ABz, Jz, 2t), M(STz, Iz, t), M(ABz, Iz, t), M(STz, Jz, t), M(Iz, Jz, t)\}$$
$$\geq M(z, STz, t),$$

giving $z = STz = Tz$ by using Lemma 3.

Therefore z is a common fixed point of AB, I, ST and J.

Case II : Now suppose that AB is continuous so that the sequences $\{(AB)^2 x_{2n}\}$ and $\{ABIx_{2n}\}$ converge to ABz. Since (AB, I) are compatible, therefore

$$M(IABx_{2n}, ABz, t) \geq \{M(IABx_{2n}, ABIx_{2n}, t/2), M(ABIx_{2n}, ABz, t/2)\}$$
$$> 1 - \lambda \text{ for all } n \geq n_0$$

Hence $IABx_{2n} \to ABz$. (1.11)

Using (1.3), (1.11) and letting $n \to \infty$, we get

$M(ABz, z, kt) \geq M(AB, z, t)$.

Therefore by Lemma 3, we have $ABz = z$.

As earlier there exists z' in X such that $ABz = z = Jz'$. Then

$$M((AB)^2 x_{2n}, STz', kt) \geq \min \{M(AB)^2 x_{2n}, Jz', 2t), M(STz', IABx_{2n}, t), M((AB)^2 x_{2n}, IABx_{2n}, t), M(STz', Jz', t), M(IABx_{2n}, Jz', t)\}$$

which on taking the limit $n \to \infty$ reduces to

$$M(z, STz', kt) \geq M(z, STz', t).$$

By Lemma 3, this implies that $STz' = z = Jz'$.

Thus z' is the coincidence point of (ST, J). Since the pair (ST, J) are R-weakly

commuting, then $STz = Jz$.

Further $M(ABx_{2n}, STz, kt) \geq \min\{M(ABx_{2n}, Jz, 2t), M(STz, Ix_{2n}, t), M(ABx_{2n}, Ix_{2n}, t),$

$$M(STz, Jz, t), M(Ix_{2n}, Jz, t)\}$$

which on letting $n \to \infty$ reduces to

$M(z, STz, kt) \geq M(z, STz, t)$ gives

$STz = z = Jz$.

Since $ST(X) \subset I(X)$, there exists a point z'' in X such that

$Iz'' = STz = z$, then

$$M(ABz'', z, kt) = M(ABz'', STz, kt)$$

$$\geq \min\{M(ABz'', Jz, 2t), M(STz, Iz'', t), M(ABz'', Iz'', t), M(STz, Jz, t), M(Iz'' Jz, t)\}$$

$$\geq M(ABz'', z, t)$$

which gives $ABz'' = z$.

Also (AB, I) are compatible and hence weakly commuting, we obtain,

$$M(ABz, Iz, t) = M(ABz'', IABz'', t) \geq M(ABz'', Iz, t)$$

$$= M(z, z, t) \to 1.$$

Therefore $ABz = Iz = z$.

Thus z is a common fixed point of AB, ST, I and J.

If the mappings ST or J is continuous instead of AB or I, then the proof that z is a common fixed point of AB, ST, I and J is similar.

Let w be another fixed point of I, J, AB and ST then

$$M(z, w, kt) = M(ABz, STw, kt)$$

$$\geq \min \{M(ABz, Jw, 2t), M(STw, Iz, t), M(ABz, Iz, t),$$

$$M(STw, Jw, t), M(Iz, Jw, t)\}$$

which gives $M(z, w, kt) \geq M(z, w, t)$

Hence $z = w$.

Finally, we need to prove that z is also a common fixed point of A, B, S, T, I and J.

Using commutativity of (A, B) and (S, T), it is easy to prove that

$Az = z = Bz$ and $Sz = z = Tz$.

Now, $M(Az, Sz, kt) = M(A\,(ABz), S\,(STz), kt)$

$$= M(A\,(BAz), S\,(TSz), kt)$$

$$= M(AB\,(Az), ST\,(Sz), kt)$$

$$\geq \min \{M(AB\,(Az), JSz, 2t), M(STSz, IAz, t), M(ABAz, IAz, t),$$

$$M(STSz, JSz, t), M(IAz, JSz, t)\},$$

Using commutativity of (A, B), (S, T), (I, A) and (J, S) we have

$M(Az, Sz, kt) \to 1$ so $Az = Sz$. Similarly it can be shown that $Bz = Tz$. Thus z is the unique common fixed point of A, B, S, T, I and J.

Remark 1. Theorem 1 remains true if we replace $a * b$ by $a.b$.

If we put $I = J$, $A = S$ and $B = T$ = Identity map in Theorem 1, we get the following result.

Corollary . Let A and J be self maps of a complete fuzzy metric space $(X, M, *)$ with continuous t-norm defined by $a * b = \min(a, b)$, $a, b \in (0, 1)$ such that (A, J) are compatible, J is continuous and

$$M(Ax, Ay, kt) \geq \min \{M(Ax, Jy, 2t), M(Ay, Jx, t), M(Ax, Jx, t), M(Ay, Jy, t),$$

$M(Jx, Jy, t)\}$

for all x, y in X, $t > 0$ and $k \in (0, 1)$.

If for all x, y in X, $\lim (x, y, t) \to 1$ as $t \to \infty$, then A and J have a unique common fixed point.

Remark 2. A number of fixed point theorems may be obtained for two to four mappings in metric, probabilistic and fuzzy metric spaces as the special cases from Theorem 3.1.

Example. Let $X = [0, 1]$. For $a, b \in [0, 1]$ define $a * b = \min(a, b)$ and

$M(x, y, t) = [\exp(|x-y|)/t]^{-1}$ for all $x, y \in X$ and $t > 0$. Then clearly $(X, M, *)$ is a fuzzy metric space (see example 2.7 in [8]).

Define mappings A, B, S, T, I and J as

$Ax = x/64$, $Bx = x/16$, $Sx = x/32$, $Tx = x/8$, $Ix = x/4$, $Jx = x$ for all $x \in [0, 1]$.

Then $AB(X) = [0, 1/1024] \subset [0, 1] = J(X)$ and $ST(X) = [0, 1/256] \subset [0, 1/4] = I(X)$.

If $\lim x_n = 0$ where $\{x_n\}$ is a sequence in X, then clearly (AB, I) are compatible.

Also (ST, J) are R-weakly commuting.

Now if we take $1/256 \le k \le 1$, $t > 0$, $x, y \in [0, 1]$ then condition (1.3) of Theorem 1 is also satisfied. Here zero is the unique common fixed point of A, B, S, T, I and J.

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KINETICS AND MECHANISM OF Mn (II) CATALYSED PERIODATE OXIDATION OF O-TOLUIDINE

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ABSTRACT

Kinetic-mechanistic studies made for periodate oxidation of o-toluidine (OMA) in acetonewater medium, have been used for derivation of rate law and proposing mechanism which satisfy various observations like first order in each reactant and catalyst, stoichiometry (1 mol OMA : 2 mol periodate), thermodynamic parameters, main product identified (Methyl-1,4 - benzoquinone), the Michaelis-Menten type kinetics being followed with respect to both reactants, the rate - pH profile, increase in rate with the increase in dielectric constant of the medium, negative primary linear type effect of ionic strength and no effect of free radical scavengers on the reaction rate.

Key words and phrases: Kinetics and mechanism, Periodate oxidation, o-toluidine. Mn (II) catalyst, Methyl-1, 4- benzoquinone.

INTRODUCTION

Reports on kinetic- mechanistic studies on the Mn (II) catalysed periodate oxidation of aromatic amines are very few [3-8] In continuation to our earlier communications on kinetic-mechanistic studies on uncatalysed periodate oxidation of few aromatic amines [9-14] and Mn (II) catalysed oxidation of 4 - chloro-2-methylaniline [15], we are reporting the kinetic-mechanistic studies on Mn (II) catalysed periodate oxidation of o-toluidine in acetone-water medium in present paper.

MATERIALS AND METHODS

Chemicals of E. Merck/CDH A.R. grade were used after distillation/ recrystallization. Triply distilled water was used for preparation of the solutions. The progress of the reaction was followed spectrophotometrically by recording the absorbance at 490 nm, the λ_{max} of reaction mixture in the duration in which the λ_{max} did not change. The pH was maintained at 4.5 by using Thiel, Schultz and Koch buffer [1] in all kinetic runs except when the effect of pH was studied.

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NaCl solutions were used for maintainning the ionic strength (μ) in the kinetic runs. Plain mirror method and Guggenheim's method were used for evaluation of initial rates [$(dA/dt)_i$] and pseudo first order rate constant k_1 (or second order rate constant k_2) respectively.

RESULTS AND DISCUSSION

Initially the reaction mixture was yellow in color, which changed into orange- wine color in about 2 minutes. On standing overnight, it changed to brown color followed by precipitation. The reaction mixture was prepared by taking periodate in excess and kept overnight. After precipitation, the supernatant liquid was extracted with petroleum ether (40-60^{0}C). On evaporation of the solvent, a yellow colored compound with melting point 68^{0}C was obtained. This compound responded positively to the test for quinone.

The UV-VIS spectrum of this compound in ethanol showed λ_{max} at 240 and 315 nm. The melting point and λ_{max} are in good agreement with the values reported for methy1-1,4-benzoquinone [2, 18]. The I.R. spectrum of this compound in KBr showed the presence of bands at 3054 cm^{-1} (s) (due to ring C-H strech), 2925 cm^{-1} (s), 2854 cm^{-1} (s) (due to C-H streching vibrations of CH_3), 1709 cm^{-1} (w) & 1744 cm^{-1} (w) (due to overtones and combination bands), 1632 cm^{-1} (s) (indicating the presence of C = O on 1, 4-benzoquinone pattern with the possibility that the position of this band gets lowered due to + I effect of alky1 group[19]), 1514 & 1457 cm^{-1} (s) (due to C = C ring strech), 1330 & 1135 cm^{-1} (m) (due to in-plane C - H bending in the ring), 751 & 664 cm^{-1} (s) (due to out-of-plane C = C and = C - H bending modes and substitution pattern in the ring). NMR spectrum in $CDC1_3$ showed the peaks at δ = 7.091, D, 2H; at δ=6.658, S, IH; and at δ = 2.156, S, 3H. The singlet at δ = 2.156 may be due to protons of - CH_3 group while other singals are due to 3 protons of the ring [19].

On the basis of melting point[Literature: M.P. = 69^{0}C, yellow plates or needles] [2] and spectral studies[Literature: UV-VIS λ_{max} at 246 nm (log $\in$= 4.14), at 312 nm (log $\in$ = 2.77), at 429 nm (log $\in$ = 1.28) and at 438 nm (log $\in$ = 1.27)], the characterstic I.R. peaks and NMR signals matching the structure of methyl-1,4- benzoquinone [18-19], this compound may be methyl 1,4-benzoquinone which is the main product of the reaction under consideration.

1 mol CMA consumed 2 moles of periodate as determined iodometrically. Tht data (table-1) indicated 2nd order for the reaction, being first order in each reactant. Linear relation between concentration of the reactants and rate supported the 2nd order kinetics. In pseudo first order conditions (table-2), the $[(dA/dt)]^{-1}$ or k_1^{-1} vs $[S]^{-1}$ plots were linear with almost negligible intercept, suggesting the Michaelis-Menten type kinetics being followed with respect to both reactants with the possibility of formation of a fast decaying intermediate complex between reactants [15-16].

Data in table-3 established the 1st order in catalyst. Rate-pH profile showed a maxima at pH= 4.5 (table-4, Fig.-1). A linear relation between log $(dA/dt)_i$ or log k_2 and 1/D with negative slope (where D is the dielectric constant of the medium) and a primary linear type plot between long $(dA/dt)_i$ or log k_2 vs ionic strength (μ) that were obtained from the data in table-5, indicated an ion-dipole interaction in this reaction. This is supported by our observation that free radical scavengers like acryl amide and allyl alcohol exerted no effect on reaction rate.

Arrhenius plot was made between 30 $\pm$ 0.1°C to 45 $\pm$ 0.1°C and the values of different thermodynamic parameters evaluated taking [OMA] = 5.0×10^{-4} M, $[NaIO_4] = 5.0 \times 10^{-3}$ M, $[Mn^{++}] = 4.0 \times 10^{-6}$ M, pH = 4.5 and acetone = 10.0% (v/v), are Ea = 8.10 k.cal mol^{-1} ; A = 4.88×10^3 lit. mol^{-1}. sec^{-1}; $\Delta S^{\neq}$=- 60.17 e. u.; $\Delta F^{\neq}$ = 18.35 k. cal. mol^{-1} and $\Delta H^{\neq}$=7.49 k.cal. mol^{-1}. Low value of energy of activation and high frequency factor are characteristic of a bimolecular reaction in the solution in which the reacting species are large. A large negative value of $\Delta S^{\neq}$ suggests the formation of strongly solvated, charged and rigid transition state.

Table-1. *Determination of order w.r. t. reactants.*

λ_{max} = 475 nm; pH= 4.5; Acetone* = 7.5% ; Acetone** = 10.0% (v/v); Temp = 35 $\pm$ 0.1°C; $[Mn^{++}] = 8.0 \times 10^{-6}$ M.

$[OMA] \times 10^3$ M	1.0*	2.0*	3.0*	5.0*	6.0*	1.0**	1.0**	1.0**	1.0**	1.0**
$[NaIO_4] \times 10^3$ M	1.0	1.0	1.0	1.0	1.0	2.0	3.0	4.0	6.0	7.0
$[dA/dt]i \times 10^2$ (min^{-1})	4.8	9.7	14.3	23.8	28.7	7.4	10.8	14.3	21.5	25.2

Table-2. *Variation of OMA and periodate*

λ_{max} = 475 nm; pH = 4.5; Acetone = 10.0% (v/v); Temp = 35.0 ± 0.1°C;

$[Mn^{++*}] = 4.0 \times 10^{-6}$ M; $[Mn^{++}]^{**} = 8.0 \times 10^{-6}$ M.

$[OMA] \times 10^3$ M	10.0*	12.0*	14.0*	18.0*	0.5**	0.5**	0.5**	0.5**	0.5**
$[NaIO_4] \times 10^3$ M	1.0	1.0	1.0	1.0	5.0	7.0	8.0	9.0	10.0
$(dA/dt)i \times 10^2$ (min^{-1})	4.8	5.8	6.9	8.7	3.8	4.4	4.8	5.6	6.3
$k_1 \times 10^3$ (sec^{-1})	3.84	4.60	5.37	6.91	4.03	5.57	6.14	6.91	7.68
$k_2 \times 10^2$ ($L.mol^{-1}.\ sec^{-1}$)	3.84	3.83	3.83	3.84	8.06	7.95	7.68	7.68	7.68
Average k_2	38.36×10^{-2}				78.08×10^{-2}				

Table-3. *Determinationof order w. r. t.* Mn^{++}.

λ_{max} = 475 nm; pH= 4.5; $[NaIO_4] = 5.0 \times 10^{-3}$ M; $[OMA] = 5.0 \times 10^{-4}$ M; Temp. = 35.0 ± 0.1°C; Acetone = 15.0 % (v/v).

$[Mn^{++}] \times 10^6$ M	4.0	5.0	6.0	70	8.0	9.0	10.0
$(dA/dt)_i \times 10^2$ (min^{-1})	2.6	3.2	3.9	4.6	5.2	5.9	6.5

Table-4. *Effect of pH on reaction rate.*

λ_{max} = 475 nm; [OMA] = 1.0×10^{-3} M; $[NaIO_4] = 1.0 \times 10^{-2}$ M; Acetone = 10.0% (v/v); Temp. = 35.0 ± 0.1°C; $[Mn^{++}] = 4.0 \times 10^{-6}$ M

pH	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0
$(dA/dt)i \times 10^{-3}$ (min^{-1})	0.72	1.92	50.0	86.0	52.0	34.0	20.0	7.6	4.4	2.8	2.6	1.1	0.71

Table-5. *Effect of Dielectric constant of medium (D) and ionic strength (μ) on the reaction rate*

λ_{max} = 475 nm; pH=4.5; [OMA] = 5.0 x 10^{-4} M; [$NaIO_4$] = 5.0 x 10^{-3} M;* Acetone = 5.0 % (v/v); Temp. = 35.0 ± 0.1° C; [Mn^{++}] = 4.0 x 10^{-6} M

D	71.0	69.7	68.4	66.8	--	--	--	--
* μ x 10^3	--	--	--	--	12.0	18.0	26.0	34.0
$(dA/dt)_i$ 10^3 (min^{-1})	44.0	36.0	31.0	26.0	74.0	59.0	45.0	34.0
k_1 x 10^4 (sec^{-1})	53.74	44.14	36.46	26.86	59.49	53.74	47.98	42.22
k_2 x 10^1 (L. mol^{-1}. sec^{-1})	10.75	8.83	7.29	5.37	20.76	20.31	19.82	19.27

The increase in the rate from pH 3.0 to 4.5 may be due to the decrease in the protonation of OMA from pH 3.0 to 4.5 (table-4, Fig.-1), which makes greater concentrations of OMA available for the reaction. This leads to the assumption that unprotonated OMA is the reactive species in present case. Second part of this profile suggests that out of various species in which periodate exists, periodate monoanion [IO_4^-] is the reactive species in present investigation [17]. Pavolva et al. [17] established that [IO_4^-] goes on decreasing with increase in pH beyound the value 4.5 decreasing thereby the rate of reaction beyond pH 4.5 Similar rate- pH profiles in case of periodate oxidation of some other aromatic amines [9-15,20] have been reported earlier by us.

Based on these results, the proposed mechanism (chart) might involve the lone pair of electrons on nitrogen atom of OMA for the co-ordinate bond formation between OMA and Mn^{++} species in a reversible step to form complex C_1 in step (i). C_1, in turn, interacts with IO_4^- in slow and rate determining step (ii) to give C_2 which changes by fast hydrolysis into C_3. The formation of a charged intermediate complex C_2 by the attack of IO_4^- on the nitrogen of anilino group and stabilization of positive charge on nitrogen of this group, has already been established and supported by LFER studies for the uncatalysed periodate oxidation of few aromatic amines [21]. In addition, a high negative value of entropy of activation and the effect of dielectric constant on the reaction rate support the involvement of solvation effects in this reaction.

It should also be noted that the initial part of the reaction is significant in the present case and the second molecule of IO_4^- reacting later to give C_4 is not significant. C_4 changes by fast hydrolysis to give C_5 i.e. the main product of reaction that has been isolated, separated and characterized in this case as methyl-1, 4-benzoquinone. The overall process may be represented as follows:

$$OMA + Mn^{++} \underset{k_2}{\overset{k_1}{\rightleftharpoons}} C_1 \qquad \text{(i)}$$

$$C_1 + [IO_4] \xrightarrow[\text{Slow}]{k_3} C_2 + Mn^{++} \qquad \text{(ii)}$$

$$C_2 \xrightarrow[\text{Fast}]{H^+, H_2O} C_3 + H_2O + HIO_3 \qquad \text{(iii)}$$

$$C_3 + IO_4^- \xrightarrow{+H^+} C_4 + H_2O + HIO_3 \qquad \text{(iv)}$$

$$C_4 \xrightarrow[\text{Fast}]{+H_2O} c_5 \text{ (Methyl-1, 4-benzoquinone)} \qquad \text{(v)}$$

On applying steady state treatment to C_1, the rate law in terms of rate of loss of $[IO_4^-]$ may be derived as follows:

Rate of loss of $[IO_4^-]$ or $- d[IO_4^-]/dt = k_3 [C_1][IO_4^-]$ = Rate of loss of C_1

$$\text{or } -\frac{d[C_1]}{dt} \qquad \text{------(1)}$$

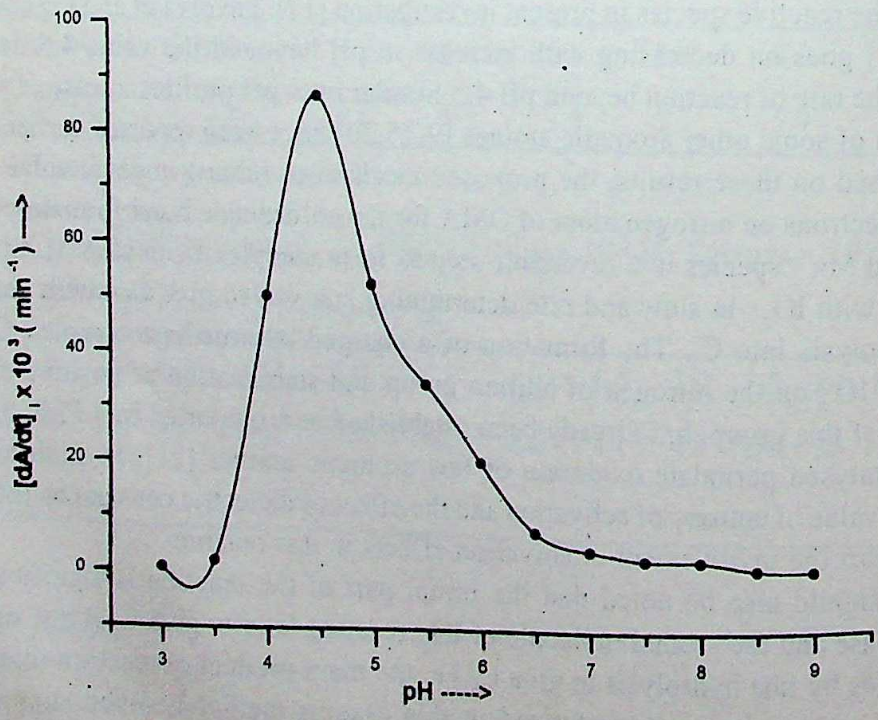
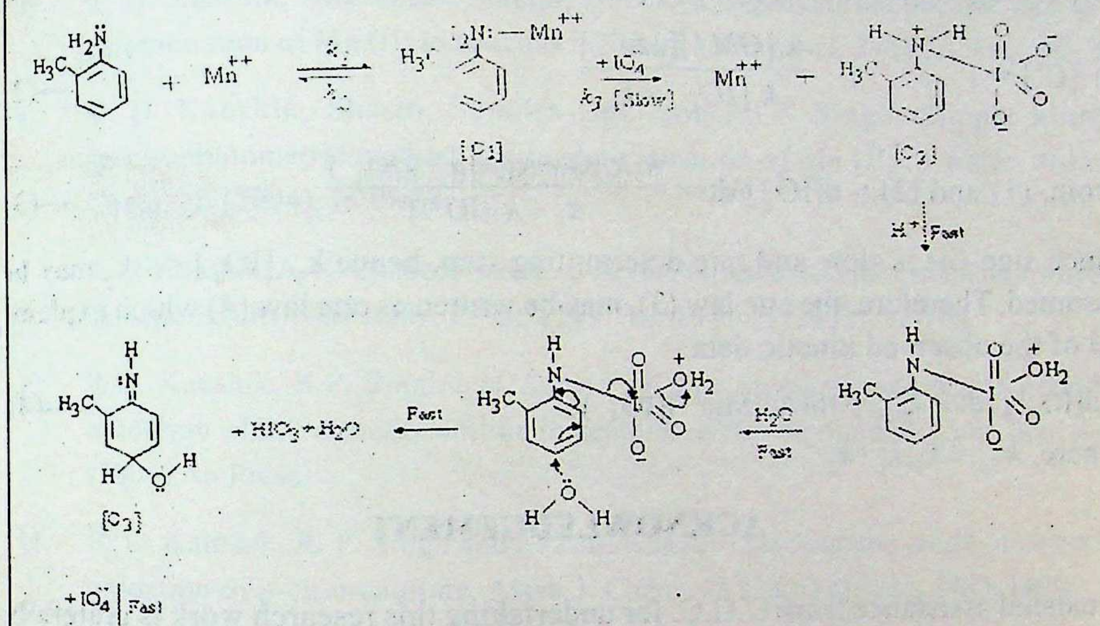


Fig. 1- Rate - pH profile

CHART - I



Methyl-1, 4- benzoquinone

Rate of formation of $C_1 = + d[C_1]/dt = k_1 [OMA][Mn^{++}] - k_2 [C_1]$

At steady state, $- d[C_1]/dt = + d[C_1]/dt$

Therefore, $k_3 [C_1][IO_4^-] = k_1 [OMA][Mn^{++}] - k_2 [C_1]$

Or $[C_1]\{k_3[IO_4^-]+k_2\} = k_1 [OMA][Mn^{++}]$

$$\text{Or } [C_1] = \frac{k_1[OMA][Mn^{++}]}{k_3[IO_4^-]+k_2} \qquad \text{---(2)}$$

$$\text{From, (1) and (2), } - d[IO_4^-]/dt = \frac{k_3 k_1 [OMA][Mn^{++}][IO_4^-]}{k_2 + k_3[IO_4^-]} \qquad \text{---(3)}$$

Since step (ii) is slow and rate determining step, hence $k_3 . [IO_4^-] << k_2$ may be assumed. Therefore, the rate law (3), may be written as rate law (4) which explains all of the observed kinetic data.

$$-d[IO_4^-]/dt = k_{obs}.[OMA][Mn^{++}][IO_4^-] \qquad \text{----(4)}$$

where, $k_{obs} = k_3 . k_1 / k_2$

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A NOTE ON GENERALIZATIONS OF MIDPOINT INEQUALITY

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Abstract

In this note we establish two new generalizations of the midpoint inequality involving two functions and their second order derivatives by using a fairly elementary analysis.

Mathematics Subject classification (1991) : 26 D 15, 41 A 55

Key words and phrases : Midpoint inequality, second order derivatives, integration by parts, Hölder's inequality.

INTRODUCTION

The following classical inequality is well known in the literature as the midpoint inequality :

$$\left|\int_a^b f(x)dx-(b-a)f\left(\frac{a+b}{2}\right)\right| \leq \frac{\|f''\|_\infty}{24}(b-a)^3, \tag{1}$$

where the mapping $f:[a,b]\subset R\to R$ is twice differentiable on the interval (a,b) and having the second derivative bounded on (a,b), *i.e.* $\|f''\|_\infty = \sup_{x\in(a,b)}|f''(x)|<\infty$.

Recently, Dragomir, et al. [1] have established the following inequality containing an integral inequality (1).

Let $f:[a,b]\subset R\to R$ be twice differentiable mapping on (a,b) and $f'':(a,b)\to R$ is integrable on (a,b). Then

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$$\left|\int_a^b f(x)dx-(b-a)f\left(\frac{a+b}{2}\right)\right| \le \begin{cases} \dfrac{(b-a)^3}{24}\|f''\|_\infty, & \text{if } f''\in L_\infty(a,b), \\ \dfrac{(b-a)^{2+\frac{1}{p}}}{8(2p+1)^{\frac{1}{p}}}\|f''\|_q, & \text{if } f''\in L_q(a,b) \\ & \text{where } \dfrac{1}{p}+\dfrac{1}{q}=1, p>1, \\ \dfrac{(b-a)^2}{8}\|f''\|_1, & \text{if } f''\in L_1(a,b). \end{cases} \tag{2}$$

For other results related to the midpoint inequality, see Chapter XV of the book by Mitrinovic et al. [2] The main purpose of this note is to establish two new generalizations of the midpoint inequality involving two functions and their second order derivatives. The analysis used in the proofs is elementary and our results provide new estimates on inequalities of this type.

STATEMENT OF RUESLTS

Our main results are established in the following theorems.

THEOREM 1. *Let $f,g:[a,b]\subset R\to R$ be twice differentiable mappings on (a,b). Suppose that $f'',g'':(a,b)\to R$ are integrable on (a,b). Then*

$$\left|2\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)-(b-a)\left[f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx+g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx\right]\right|$$

$$\le \begin{cases} \dfrac{(b-a)^3}{24}\left[\|f''\|_\infty\displaystyle\int_a^b|g(x)|dx+\|g''\|_\infty\int_a^b|f(x)|dx\right], & \text{if } f'',g''\in L_\infty(a,b), \\ \dfrac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left[\|f''\|_p\displaystyle\int_a^b|g(x)|dx+\|g''\|_p\int_a^b|f(x)|dx\right], & \text{if } f'',g''\in L_p(a,b), \\ & \text{where } \dfrac{1}{p}+\dfrac{1}{q}=1, q>1, \\ \dfrac{(b-a)^2}{8}\left[\|f''\|_1\displaystyle\int_a^b|g(x)|dx+\|g''\|_1\int_a^b|f(x)|dx\right], & \text{if } f'',g''\in L_1(a,b). \end{cases} \tag{3}$$

Theorem 2. Let f, g, f'', g'' be as in Theorem 1. Then

$$\left| g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx + f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx - 2(b-a)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \right|$$

$$\leq \begin{cases} \dfrac{(b-a)^3}{24}\left[\|f''\|_\infty \left|g\left(\dfrac{a+b}{2}\right)\right| + \|g''\|_\infty \left|f\left(\dfrac{a+b}{2}\right)\right|\right], \text{ if } f'', g'' \in L_\infty(a,b), \\ \dfrac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left[\|f''\|_p \left|g\left(\dfrac{a+b}{2}\right)\right| + \|g''\|_p \left|f\left(\dfrac{a+b}{2}\right)\right|\right], \text{ if } f'', g'' \in L_p(a,b), \\ \qquad\qquad \text{where } \dfrac{1}{p}+\dfrac{1}{q}=1, q>1, \\ \dfrac{(b-a)^2}{8}\left[\|f''\|_1 \left|g\left(\dfrac{a+b}{2}\right)\right| + \|g''\|_1 \left|f\left(\dfrac{a+b}{2}\right)\right|\right], \text{ if } f'', g'' G L_1(a,b), \end{cases} \quad (4)$$

PROOFS OF THEOREMS 1 AND 2

Form the hypotheses and using the integration by parts formula, the following identities hold $(\text{See}[1, p.26])$:

$$\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) = \int_a^b k(x)f''(x)dx, \quad (5)$$

$$\int_a^b g(x)dx - (b-a)g\left(\frac{a+b}{2}\right) = \int_a^b k(x)g''(x)dx, \quad (6)$$

where

$$k(x) = \begin{cases} \dfrac{(x-a)^2}{2} & x \in \left[a, \dfrac{a+b}{2}\right] \\ \dfrac{(b-x)^2}{2} & x \in \left[\dfrac{(a+b)}{2}, b\right]. \end{cases} \quad (7)$$

Multiplying both sides of (5) and (6) by $\int_a^b g(x)dx$ and $\int_a^b f(x)dx$ respectively and and adding the resulting identities we have

$$2\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)-(b-a)f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx -(b-a)g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx \tag{8}$$

$$=\left(\int_a^b g(x)dx\right)\left(\int_a^b k(x)f''(x)dx\right)+\left(\int_a^b f(x)dx\right)\left(\int_a^b k(x)g''(x)dx\right).$$

From (8) and using the properties of modulus we have

$$\left|2\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)-(b-a)\left[f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx+g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx\right]\right|$$

$$\le\left(\int_a^b |g(x)|dx\right)\left(\int_a^b |k(x)||f''(x)|dx\right)+\left(\int_a^b |f(x)|dx\right)\left(\int_a^b |k(x)||g''(x)|dx\right). \tag{9}$$

We consider the following three cases.

(i) If $f'', g'' \in L_\infty(a,b)$, then

$$\int_a^b |k(x)||f''(x)|dx \le \|f''\|_\infty \int_a^b |k(x)|dx$$

$$=\|f''\|_\infty\left[\int_a^{\frac{a+b}{2}} \frac{(x-a)^2}{2}dx+\int_{\frac{a+b}{2}}^b \frac{(b-x)^2}{2}dx\right]=\frac{(b-a)^3}{24}\|f''\|_\infty. \tag{10}$$

Similarly,

$$\int_a^b |k(x)||g''(x)|dx \le \frac{(b-a)^3}{24}\|g''\|_\infty. \tag{11}$$

Using (10) and (11) in (9) we get the first inequality in (3).

(ii) If $f'', g'' \in L_p(a,b)$ where $\frac{1}{p}+\frac{1}{q}=1, q>1$, then

$$\int_a^b |k(x)||f''(x)|dx \le \|f''\|_p \left(\int_a^b |k(x)|^q dx\right)^{\frac{1}{q}}$$

$$= \|f''\|_p \left(\int_a^{\frac{a+b}{2}} \left(\frac{(x-a)^2}{2}\right)^q dx + \int_{\frac{a+b}{2}}^b \left(\frac{(b-x)^2}{2}\right)^q dx\right)^{\frac{1}{q}} = \|f''\|_p \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}. \qquad (12)$$

Similarly,

$$\int_a^b |k(x)||g''(x)|dx \le \|g''\|_p \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}. \qquad (13)$$

Using (12) and (13) in (9), the second inequality in (3) holds.

(iii) If $f'', g'' \in L_1(a,b)$, then

$$\int_a^b |k(x)||f''(x)|dx \le \max_{x\in(a,b)} k(x)\|f''\|_1 = \frac{(b-a)^2}{8}\|f''\|_1. \qquad (14)$$

Similarly,

$$\int_a^b |k(x)||g''(x)|dx \le \frac{(b-a)^2}{8}\|g''\|_1. \qquad (15)$$

Using (14) and (15) in (9), we get the last inequality in (3). The proof of Theorem 1 is complete. In order to prove Theorem 2, we multiply both sides of (5) and (6) by

$g\left(\frac{a+b}{2}\right)$ and $f\left(\frac{a+b}{2}\right)$ respectively and add the resulting identities to get

$$g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx + f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx - 2(b-a)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$

$$= g\left(\frac{a+b}{2}\right)\int_a^b k(x)f''(x)dx + f\left(\frac{a+b}{2}\right)\int_a^b k(x)g''(x)dx. \tag{16}$$

From (16) and using the properties of modulus we have

$$= \left| g\left(\frac{a+b}{2}\right)\int_a^b f(x)dx + f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx - 2(b-a)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \right|$$

$$\leq \left| g\left(\frac{a+b}{2}\right)\right| \int_a^b |k(x)||f''(x)|dx + \left| f\left(\frac{a+b}{2}\right)\right| \int_a^b |k(x)||g''(x)|dx \tag{17}$$

The rest of the proof can be completed by following the last arguments as in the proof of Theorem 1 below the inquality (9).

In concluding, we note that, in the special case, If we take $g(x) = 1$ and hence $g''(x) = 0$ in Theorems 1 and 2, then we get the inequality (2) which in turn contains the well known midpoint inaquality given in (1).

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COMMON FIXED POINT THEOREMS FOR FOUR SELF-MAPS ON METRIC SPACES BY ALTERING DISTANCES

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ABSTRACT

We obtain some common fixed point theorems for four self maps on metric spaces by alteration of distances using various definitions and conditions.

Key words: Fixed point, concidence point, partialy commuting, f-continuous, f-compatible, reciprocally continuous, altering distances.

2000 Mathematics subject classification : 47H10, 54H25.

INTRODUCTION

We prove some fixed point theorems for four self-maps by altering distances using various definitions and conditions.

First we give some well-known definitions.

Through out this paper, unless otherwise stated, $R(f)$ is the range of f.

DEFINITION 1. (Naidu [6]) The pair (f, g) is said to be f-continuous at z if $\{ffx_n\}$ and $\{fgx_n\}$ converge to fz whenever $\{x_n\}$ is a sequence in X such that $\{fx_n\}$ and $\{gx_n\}$ converge to z.

DEFINITION 2. (Jessy[2]) The ordered pair (f,g) is said to be weak comptaible at z if $\{gfx_n\}$ converges to fz whenever $\{x_n\}$ is a sequence in X such that $\{fx_n\}$ and $\{gx_n\}$ converge to z, and $\{fgx_n\}$ and $\{ffx_n\}$ converge to fz.

DEFINITION 3. (G.Jungck [4]) The pair (f,g) is said to be partially commuting at z or weakly compatible at z if $fz = gz$ provided there exists $w \in X$ such that $fw = gw = z$.

DEFINITION 4. (Pathak and Khan [8]) The pair (f,g) is said to be f-compatible if $d(fgx_n, ggx_n) \to 0$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence in X such that $fx_n \to z$ and $gx_n \to z$ as $n \to \infty$.

DENINITION 5. {Jungck [3]) The pair (f,g) is said to be compatible (also called asymptotically commuting by Tiwari & Singh [15] see also [1] & [14]) if $(fgx_n, gfx_n) \to 0$ as $n \to \infty$ whenever $\{x_n\}$is a sequence in X such that $fx_n \to z$ and

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$gx_n \to z$ *as* $n \to \infty$.

DEFINITION 6. (Pant [7]) The pair (f,g) is said to be reciprocally continuous on X if $\{fgx_n\}$ converges to fz and $\{gfx_n\}$ converge to gz whenever $\{x_n\}$ is a sequence in X such that $fx_n \to z$ and $gx_n \to z$ as $n \to \infty$.

DEFINITION 7. (Naidu[6]) The ordered pair (f,g) is said to be weakly f-compatible at z if either $\{gfx_n\}$ or $\{ggx_n\}$ converges to fz whenever $\{x_n\}$ is a sequence in X such that $\{fx_n\}$ and $\{gx_n\}$ converge to z, and $\{fgx_n\}$ and $\{ffx_n\}$ converge to fz.

Naidu [6] observed the following

REMARK 8.

I. If f is continuous at z, then (f,g) is f-continuous at z.

2. If (f, g) is compatible at z, then (f,g) is weak compatible at z.

3. If (f,g) is either weak compatible or f-compatible or g-compatible, then $(f,g,)$ is weakly f-compatible at z for any $z \in X$.

4. If (f,g) is weakly f-compatible at z, then it is partially commuting at z.

MAIN RESULTS

THEOREM 9. Let (X,d) be a metric space and $f,g, S,T: X \to X$ be such that

$$\begin{aligned}\psi(d(fx,gy)) \le \varphi(\max\{&\psi(d(Sx,Ty)) + \psi(d(fx,Sx) + \psi(d(gy,Ty)),\\ &\psi(d(fx,Sx)) + \psi(dSx,gy)), \psi(d(gy,Ty)) + \psi(d(fx,Ty))\})\end{aligned} \tag{9.1}$$

for all $x, y \in X$, where $\varphi\ \psi : \Re^+ \to \Re^+$, φ is monotonically incerasing on $\Re^+$, $\varphi(t+) < t$ for all $t > 0$, ψ is continuous and monotonically incereasing on $\Re^+$ and $\psi(t)=0 \Leftrightarrow t=0$.

Suppose that there is a sequence $\{x_n\}$ in X such that

$fx_{2n} = Tx_{2n+1}$ ($= y_{2n,}$ say)

$gx_{2n+1} = Sx_{2n+2}$ ($= y_{2n+1,}$ say) for $n = 0, 1, 2,$........

and $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$. (9.2)

Then $\{y_n\}$ is a Cauchy sequence in X.

Further assume that $\{y_n\}$ converges to some $z \in X$.

Then the following statements are true.

1. If $u \in X$ and $Su = fu = u$, then $u = z$.

2. If $v \in X$ and $Tv = gv = v$, then $v = z$.
3. If $Sz = fz$, then $fz, = Sz = z$.
4. If $Tz = gz$, then $gz = Tz = z$.
5. If $Sz = z$, then $fz = z$.
6. If $Tz = z$, then $gz = z$.
7. If (f, S) is f-continuous at z and weakly f-compatible at z, then $fz = z$.
8. If (g, T) is g-continuous at z and weakly g-compatible at z, then $gz = z$.
9. If $z \in R(S)$ and (f,S) is partially commuting at z, then $fz = Sz = z$.
10. If $z \in R(T)$ and (g, T) is partially commuting at z, then $gz = Tz = z$.
11. If (f,S) is S-continuous at z and weakly S-compatiable at z, then $fz = Sz=z$.
12. If (g,T) is T-continuous at z and weakly T-compatiable at z, then $gz=Tz=z$..
13. If (f,S) is compatible at z and reciprocally continuous at z, then $fz = Sz=z$.
14. If (g,T) is compatible at z and reciprocally continuous at z, then $gz = Tz = z$.
15. If $fz= Sz = Tz$, then $fz = Sz = Tz = gz = z$
16. If $gz = Sz = Tz$, then $fz = Sz = Tz = gz = z$.
17. Maps f. and S have a unique common fixed point, if one of the following statements is true.
    (i) $fz = Sz$,
    (ii) (f, S) is S-continuous at z and weakly S-compatible at z,
    (iii) $z \in R(S)$ and (f,S) is partially commuting at z,
    (iv) (f, S) is compatible at z and reciprocally continuous at z.
18. Maps g and T have a unique common fixed point, if one of the follwoing statements is true.
    (i) $gz = Tz$,
    (ii) (g, T) is T-continuous at z and weakly T-compatible at z,
    (iii) $z \in R(T)$ and (g,T) is partially commuting at z.
    (iv) (g,T) is compatible at z and reciprocally continuous at z.
19. z is the unique common fixed point of f and S, as well as g and T, if at

least one of the statements (i), (ii), (iii) and (iv) of Statement 17 and at least one of those of Statement 18 is true.

20. z is the unique common fixed point of f and S, as well as g and T, if $Sz = Tz$ and at least one of the statements (i), (ii), (iii) and (iv) of statement 17 or at least one of those of Statement 18 is true.

PROOF. Suppose that $\{y_{2n}\}$ is not Cauchy.

Then, since $\{d(y_n, y_{n+1})\}$ converges to zero, there exists $\varepsilon > 0$ and strictly increasing sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that $m_k \geq n_k \geq k, \left(y_{2n_k}, y_{2n_k}\right) \geq \varepsilon$ and

$d\left(y_{2m_k}, y_{2m_k-2}\right) < \varepsilon$ for all $k = 1, 2, 3$........

we have $\varepsilon \leq d\left(y_{2n_k}, y_{2m_k}\right) \leq d\left(y_{2n_k}, y_{2m_k-2}\right) + d\left(y_{2m_k-2}, y_{2m_k-1}\right) + d\left(y_{2m_k-1}, y_{2m_k}\right)$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k}) = \varepsilon \tag{9.3}$$

Also $\lim_{k\to\infty} d(y_{2n_k+1}, y_{2m_k-1}) = \varepsilon$ and $\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k-1}) = \varepsilon$

(from (9.2), (9.3) and triangle inequality)

From (9.1) we have

$$\begin{aligned}\psi(d(fx_{2m_k}, gx_{2n_k+1})) \leq \varphi(max\{&\psi(d(Sx_{2m_k}, Tx_{2n_k+1})) + \psi(d(fx_{2m_k}, Sx_{2m_k})) + \psi(d(gx_{2n_k+1}, Tx_{2n_k+1}),\\ &\psi(d(fx_{2m_k}, Sx_{2m_k})) + \psi(Sx_{2m_k}, gx_{2n_k+1})),\\ &\psi(d(gx_{2n_k+1}, Tx_{2n_k+1})) + \psi(d(fx_{2m_k}, Tx_{2n_k+1}))\}),\end{aligned}$$

That is

$$\begin{aligned}\psi(d(y_{2m_k}, y_{2n_k+1})) \leq \varphi(\max\{&\psi(d(y_{2m_k-1}, y_{2n_k})) + \psi(d(y_{2m_k}, y_{2m_k-1})) +\\ &\psi(d(y_{2n_k+1}, y_{2m_k})), \psi(d(y_{2m_k}, y_{2m_k-1})) + \psi(d(y_{2m_k-1}, y_{2n_k+1})),\\ &\psi(d(y_{2n_k+1}, y_{2n_k})) + \psi(d(y_{2m_k}, y_{2n_k}))\}).\end{aligned}$$

Letting $k \to \infty$, we get

$$\psi(\varepsilon) \leq \varphi(\max\{\psi(\varepsilon) + \psi(0) + \psi(0), \psi(0) + \psi(\varepsilon), \psi(0) + \psi(\varepsilon)\})$$

$= \varphi(\psi(\varepsilon))$ (since $\psi(0) = 0$).

Since $\phi(t+) < t$ for all $t > 0$, we must have $\psi(\varepsilon) = 0$.

Since $\psi(t) = 0 \Rightarrow t = 0$, we have $\varepsilon = 0$.

This is a contradiction. Hence $\{(y_{2n.}\}$ is a Cauchy sequence.

Since $\{d(y_n, y_{n+1})\}$ converges to zero, it follows that $\{y_n\}$ is a Cauchy.

Suppose now that $\{y_n\}$ converges to z for some $z \in X$. From (9.1), we have

$$\psi(d(fu, gx_{2n+1})) \le \varphi(\max\{\psi(d(Su, Tx_{2n+1})) + \psi(d(fu, Su)) + \psi(d(gx_{2n+1} Tx_{2n+1})),$$
$$\psi(d(fu, Su)) + \psi(d(Su, gx_{2n+1})),$$
$$\psi(d(gx_{2n+1}, Tx_{2n+1})) + \psi(d(fu, Tx_{2n+1})),$$

Letting $n \to \infty$, we get

$$\psi(d(u,z)) \le \varphi(\max\{\psi(d(u,z)) + \psi(0) + \psi(0), \psi(0) + \psi(d(u,z)), \psi(0) + \psi(d(u,z))\})$$

(since $Su = fu = u$).

Hence $d(u, z) = 0$. Therefore $u = z$. This proves (1).

(2) Follows as in (1).

(3) Suppose that $Sz = fz$.

By taking $x = z$, $y = x_{2n+1}$ in (9.1) and letting $n \to \infty$ we get.

$$\psi(d(fz,z)) \le \varphi(\max\{\psi(d(fz,z)) + \psi(0) + \psi(0), \psi(0) + \psi(d(fz,z)),$$
$$\psi(0) + \psi(d(fz,z))\}).$$

Hence $\psi(d(fz,z)) = 0$. Hence $d(fz, z) = 0$. Therefore $fz = z$.

(4) Follows as in (3).

(5) Suppose that $Sz = z$.

By taking $x = z$, $y = x_{2n+1}$ in (9.1) and letting $n \to \infty$, we get

$$\psi(d(fz,z)) \le \varphi(\max\{\psi(0) + \psi(d(fz,z)) + \psi(0), \psi(d(fz,z)) + \psi(0),$$
$$\psi(0) + \psi(d(fz,z))\}).$$

Hence $\psi(d(fz,z)) = 0$. Hence $d(fz,z) = 0$. Therefore $fz = z$.

(6) Follows as in (5).

(7) Suppose that (f, S) is f-continuous at z and weakly f-compatible at z.

Since $fx_{2n} \to z$, $Sx_{2n} \to z$ and (f, S) is f-continuous at z, it follows that $ffx_{2n} \to fz$ and $fSx_{2n} \to fz$. Since (f, S) is weakly f-compatible at z, it follows that either $Sfx_{2n} \to fz$ or $SSx_{2n} \to fz$.

Case (i) : Suppose that $Sfx_{2n} \to fz$. From (9.1)

$$\psi(d(ffx_{2n}, gx_{2n+1})) \le \varphi(\max\{\psi(d(Sfx_{2n}, Tx_{2n+1})) + \psi(d(ffx_{2n}, Sfx_{2n}) + \psi(d(gx_{2n+1}, Tx_{2n+1})), \psi(d(ffx_{2n}, Sfx_{2n})) + \psi(d(Sfx_{2n}, gx_{2n+1})), \psi(d(gx_{2n+1}, Tx_{2n+1})) + \psi(d(ffx_{2n}, Tx_{2n+1}))\}),$$

Letting $n \to \infty$, we get.

$$\psi(d(fz, z)) \le \varphi(\max\{\psi(d(fz, z)) + \psi(0) + \psi(0), \psi(0) + \psi(d(fz, z)), \psi(0) + \psi(d(fz, z))\}).$$

Hence $\psi(d(fz, z)) = 0$ and $d(fz,z)=0$, i.e. $fz = z$.

Case (ii) : Suppose that $SSx_{2n} \to fz$.

Then by taking $x = Sx_{2n}$, $y = x_{2n+1}$ in (9.1) and letting $n \to \infty$, as in case (i) we get $fz = z$

(8) Follows as in (7).

(9) Since $z \in R(S) \exists\, w \in X$ such that $Sw = z$.

Now by taking $x = w, y = x_{2n+1}$ in (9.1) and letting $n \to \infty$ we get

$$\psi(d(fw, z)) \le \varphi(\max\{\psi(0) + \psi(d(fw, z)) + \psi(0), \psi(d(fw, z)) + \psi(0), \psi(0) + \psi(d(fw, z))\}).$$

Hence $\psi(d(fw, z)) = 0$, and $d(fw, z) = 0$ i.e. $fw = z$.

Since (f,S) is the partially commuting at z and $Sw = fw = z$, we have $fz = Sz$.

The remaining part follows from Statement (3).

(10) Follows an in (9).

(11) Suppose that (f, S) is S-continuous at z and weakly S-compatible at z.

Since $fx_{2n} \to z$ $Sx_{2n} \to z$ and (f,S) is S-continuous at z, it follows that $Sfx_{2n} \to Sz$ and

$SS_{2n} \to Sz$. Since (f,S) is weakly S-compatible at z, it follows that either $fSx_{2n} \to Sz$ or $ffx_{2n} \to Sz$.

Case (i) : Suppose that $fSx_{2n} \to Sz$

From (9.1), we have

$$\psi(d(fSx_{2n}, gx_{2n+1})) \le \varphi(\max\{\psi(d(SSx_{2n}, Tx_{2n+1})) + \psi(d(fSx_{2n}, SSx_{2n})) + \psi(d(gx_{2n+1}Tx_{2n+1})),$$

$$\psi(d(fS_{2n}, SSx_{2n})) + \psi(d(SSx_{2n}, gx_{2n+1})),\ \psi(d(gSx_{2n+1}, Tx_{2n+1})) + \psi(d(fSx_{2n}Tx_{2n+1}))\}).$$

Letting $n \to \infty$, we get.

$$\psi(d(Sz,z)) \le \varphi(\max\{\psi(d(Sz,z)) + \psi(0) + \psi(0), \psi(0) + \psi(d(Sz,z)), \psi(0) + \psi(d(Sz,z))\}).$$

Hence $\psi(d(Sz,z)) = 0$, and $d(Sz, z) = 0$, i.e. $Sz = z$.

Since $Sz = z$, $z \in R(S)$.

Since (f,S) is weakly S-compatible at z, it follows that (f, S) is partially commuting at z. Since $z \in R(S)$ and (f,S) is partially commuting at z, from Statement (9) it follows that $fz = Sz = z$.

Case (ii) Suppose that $ffx_{2n} \to Sz$.

Now by taking $x = fx_{2n}$, $y = x_{2n+1}$ in (9.1) and letting $n \to \infty$, we get $Sz = z$.

The rest of the proof follows as in Case (i).

(12) Follows as in (11).

(13) Since (f, S) is compatible, follows that $d(fSx_{2n}, Sfx_{2n}) \to 0$ as $n \to \infty$ whenever $fx_{2n} \to z$ and $Sx_{2n} \to z$ as $n \to \infty$. Since (f, S) is reciprocally continuous at z, follows that $fSx_{2n} \to fz$ and $Sfx_{2n} \to Sz$. Hence from the compatibility and reciprocal continutiy of (f,S) at z, we have $fz = Sz$.

The remaining follows from statement (3).

(14) Follows as in (13).

(15) Follows from statements (3) and (6).

(16) Follows from statements (4) and (5).

Statements (17), (18), (19) and (20) are trivial from the above proofs.

Now we state the following theorem, which is a corollary of Theorem 9.

THEOREM 10. Let (X, d) be a metric space and $f, g, S, T: X \to X$ be such that

$$\psi(d(fx, gy)) \le \varphi(\max\{\psi(d(Sx, Ty)), \psi(fx, Sx)), \psi(d(gy, Ty)),$$
$$\psi(d(fx, Ty)), \psi(d(Sx, gy))\}) \qquad (10.1)$$

for all $x, y \in X$, where $\varphi, \psi: \Re^+ \to \Re^+$, φ is monotonically increasing on $\Re^+$, $\varphi(t+) < t$ for all $t > 0$, ψ is continuous and montonically increasing on $\Re^+$ and $\psi(t) = 0 \Rightarrow t = 0$.

Suppose that there is a sequence $\{x_n\}$ in X such that

$fx_{2n} = Tx_{2n+1}$ ($= y_{2n}$, say)

$gx_{2n+1} = Sx_{2n+2}$ ($= y_{2n+1}$, say) for $n = 0, 1, 2, \ldots\ldots\ldots\ldots$

and $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$.

Then $\{y_n\}$ is a Cauchy sequence in X.

Further assume that $\{y_n\}$ converges to some $z \in X$.

Then statements (1) to (20) of Theorem 9 are true here also.

REMARK 11. Theorem 10 is a generalization of Theorem 1 of Naidu & Rajendra Prasad [5], Theorem 2.1, 2.4 of Sastry et al. [13], Theorem 1 of Sastry et al. [10], Theorem 7 and 12 of Sastry et al. [11]. Also Theorem 2 of Rathore et al. [9] and Theorem 3.1 of Sastry et al. [12] are corollaries of Theorem 10.

Finally we state a theorem which is a slight variant of Theorem 9.

THEOREM 12. Let (X,d) be a metric space and $f, g, S, T: \to X$ such that

$$\psi(d(fx, gy) \le \phi(\max\{\psi(d(Sx, Ty)), \psi(d(fx, Ty)), \psi(d(Sx, gy))\}\,.$$
$$+ \psi(d(fx, Sx)) + \psi(d(gy, Ty)))$$

for all $x, y \in X$, where $\varphi, \psi: \Re^+ \to \Re^+$, φ is monotonically increasing on $\Re^+$, $\varphi(t+) < t$ for all $t > 0$, ψ is continuous and montonically increasing on $\Re^+$ and $\psi(t) = 0 \Rightarrow t = 0$.

Suppose that there is s sequence $\{x_n\}$ in X such that

$fx_{2n} = Tx_{2n+1}$ ($= y_{2n}$, say)

$gx_{2n+1} = Sx_{2n+2}$ ($= y_{2n+1}$, say) for $n = 0, 1, 2, \ldots\ldots\ldots\ldots$

and d $(y_n, y_{n+1}) \to 0$ as $n \to \infty$.

Then $\{y_n\}$ is a Cauchy sequence in X.

Further, assume that $\{y_n\}$ converges to some $z \in X$.

Then the following statements are true.

1. If $u \in X$ and $Su = fu, = u$ then $u = z$.
2. If $v \in X$ and $Tv = gv = v$, then $v = z$.
3. If $Sz = fz$, then $fz = Sz = z$.
4. If $Tz = gz$, then $gz = Tz = z$.
5. If (f, S) is f-continuous at z and weakly f-compatible at z, then $fz = z$.
6. If (g, T) is g-continuous at z and weakly g-compatibleat z, then $gz = z$.
7. If (f, S) is S-continuous at z and weakly S-compatible a z, then $Sz = z$.
8. If (g, T) is T.continuous at z and weakly T-compatile at z, then $Tz= z$.
9. If (f,S) is compatible at z and reciprocally continuous at z, then $fz = Sz= z$.
10. If (g, T) is compatible a z and reciprocally continuous at z, then $gz = Tz = z$.
11. Maps f and S have unique common fixed point if one of the following statements is true.

    (i) $fz = Sz$

    (ii) (f,S) is comatible at z and reciprocally continuous at z.
12. Maps g and T have unique common fixed point if one of the following statemts is true.

    (i) $gz = Tz$

    (ii) (g, T) is comatible at z and reciprocally continuous at z.
13. Z is unique common fixed point of f and S, as well as g and T, if at least one of the following statement (i) and (ii) of statement (11) and at least on of those of statement (12) are true.

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HARMFUL BIOEFFECTS OF HIGH FREQUENCY ELECTROMAGNETIC FIELDS

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ABSTRACT

The influence of electromagnetic field (EMF) of different frequencies at different distances from the transmission tower is discussed Human body is a dielectric and consists of complex matrices of electric and magnetic fields. When the body is exposed to high frequency radiation, the electric field is induced inside the body. It is found that the current density at the surface of skin and bone increase with frequency of radiation being exposed High frequency radation also produces the dangerous deseases as eukemia and cancer risk in childhood.

Key works: Current density, electromagnetic field, conductivity of skin and bone

INTRODUCTION

Due to excessive modernization human beings are always surrounded by electromagnetic environment. Broadcast and communication transmitters as well as home appliances like induction heaters, VDU and TV sets and medical equipments emit EMFs of different frequencies and intensities. Though, EMFs of low frequencies are not so injurious but all types of high frequency EMFs affect the human body in many ways causing many diseases such as leukemia, brain tumor, and cancer risk in childhood [5,6,12]. These radiations affect the tissues, cells and skin of human body by induction of the electric and magnetic fields inside the body on exposure to the radiation. It seems well suited to predict changes in thermo- regulatory responses that may result from deposition of EM energy into the body and has provided the basis for several such analysis [4, 14, 16]. The EM radiation influences themoregulatory behaviour like other sources of heat Hardy and Guieu [7] has summarized its themophysiological effects of human body. Similarly, work has been done on physical character of human body in relation to heat loss [3]. If only one part of body (such as head or trunk) is exposed to EM radiation, the change in selected ambient temperature range is governed by an integrated energy deposition over the whole body and not by energy deposited in some specific locus as the brain. Indeed the hypothalamus appears to play no more vital role in behavioural thermoregulation during EM exposure than what the other thermo senstive regions of the body do [1]. Therefore interaction may be expected between the body and radiation resulting into many types of biological effects. Efforts have been made to decribe this type of interaction by Polk [11] and Mason et al. [9]. In the present paper current density in

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human skin and bone due to electromagnetic environment have been studied.

MECHANISM

The EMF produces two type of effects that control the dielectric behaviour. On is the oscillation of free charges or ions and the other is the rotation of dipole molecules at the frequencies of the applied EM energy. The first gives rise to conduction currents with an associated energy loss due to electrical resistance of the medium with an associated dielectric loss due to viscosity. Here we study the current density in the skin and human body induced due to the radiation of 10 kW transmission towers. Propagation of these rediations in the atmosphere means the propagation of electric and magnetic fields around the tower. These fields decrease as inversely proportional to distance from the tower. Simultaneously the conductivity of skin varies with frequency of radiation. If σ is the conductivity and E is the exposed electric fields, the surface current density in the skin is given by

$$\mathbf{J} = \sigma \mathbf{E}$$

The magnitude of induced electric field at a distance r from the antenna tower emitting at 10 kW is given by Pathak et al. [10]

$$E = 774.6/r$$

RESULTS AND DISCUSSION

The conductivity of skin and bone are increased with radiation frequency of exposure to body reported by Johnson and Guy [8]. Using these data calculated values of current density induced on the skin of human body at distances 100 m, 300 m, 500 m and 1 km from the tower are shown in Table 1. Table 2 shows calculated values of current density induced on bones of human body at various distance of 100 m. 300 m, 500 m and 1 km from the tower. Current flow or current density may affect the flow of ions. Watchtel [15] observed shift in neuron firing rate response to injected transmembrane. He also injected low frequency current into the seawater surrounding the cell preparation through external electrodes and observed some variation in effects. Shwan [13] proposes that current density in the tissues is the most useful parameter to the hazards. But the exact mechanism is to be worked out as to how EMF produces the heat inside the biological material, which also breaks down the protective mechanisms of heat control inside the body [2].

The values of current densities on the skin and bone of human beings are also increased with frequency of radiation as shown in Tables 1 and 2, which affect the human body in various ways. Thus the presence of transmission tower is harmful for the public health and therefore should be installed far away from populated areas.

Table 1: Variation of conductivity and current density on the skin of human beings with high frequency ratiation.

Frequency of radiation (MHz)	Conductivity (mho)	Current density at different distances from the tower			
		100m	300m	500m	1 Km.
10	0.625	0.0893	0.0298	0.0178	0.00893
27.12	0.662	0.1337	0.0445	0.0267	0.0133
40.68	0.693	0.1621	0.54	0.0323	0.1614
100	0.889	0.28	0.0934	0.056	0.028
200	1.28	0.5081	0.1694	0.1063	0.0506
300	1.37	0.5684	0.1894	0.1135	0.0567
433	1.43	0.6034	0.2031	0.12	0.06034
750	1.54	0.6622	0.02208	0.1349	0.662
915	1.6	0.7008	0.2337	0.1401	0.07008
1500	1.77	0.8053	0.2686	0.1612	0.805
2450	2.21	1.047	0.3491	0.2095	0.1045
3000	2.26	1.0938	0.3645	0.2187	0.1091
5000	3.92	1.979	0.6597	0.3959	0.1975
5900	4.73	2.421	0.8083	0.4848	0.2421
8000	7.65	4.2304	1.4106	0.846	0.4222
10000	10.3	5.7062	1.9034	1.1422	0.5706

Table 2 : Variation of conductivity and current density in the bones of human beings with high frequency ratiation.

Frequency of radiation (MHz)	Conductivity (mho)	Current density at different distances from the tower			
		100m	300m	500m	1 Km.
27.12	27.05	28.564	9.5216	5.7129	2.853
40.68	32.07	45.747	15.238	9.1527	4.5714
100	47.5	116.802	38.931	23.355	11.67
200	60	175.38	58.458	35.37	17.52
300	69.3	209.078	69.708	41.787	20.893
433	77.95	238.293	79.446	47.658	23.501
750	93.9	287.052	95.702	57.41	28.31
915	101.3	309.674	103.244	61.934	30.541
1500	120.9	369.591	123.221	73.918	36.451
2450	154.7	479.26	159.774	95.821	47.895
3000	172	532.856	177.641	106.571	53.251
5000	235.5	729.579	243.224	145.915	72.91
5900	262	863.552	287.859	172.684	85.276
8000	343	1189.52	396.542	237.87	118.849
10000	436.5	1560.48	520.133	312.053	155.917

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METHODS FOR SOLVING CUBIC EQUATIONS

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ABSTRACT

In this paper different methods for solving cubic equation are given. The contribution of Indian and Foreign mathematicians in the development for solving cubic equation are also discussed. Specially Omar Khayyam's geometrical method for solving particular type of cubic equation and Vieta's method for solving a cubic equation by converting it into quadratic are also explained.

Key Words : Cubic, Quadratic, Real, Imaginary

Classification No. : 01A32

INTRODUCTION

An equation of the form $ax^3+bx^2+cx+d=0$ where a, b, c and d are real numbers and $a \neq 0$ is called real cubic equation.

The oldest cubic equation which is of the form $x^3= k$ is due to Mehaxchumus (c. 350 B.C.) [7].

The next cubic equation is due to Archimedes (c.200 B. C.) [7].

$$x^3+c^2b=cx^2$$

Eutocius (c. 560 A.D.) solved the cubic equation of Archimedes by finding the intersection of two conics [7].

$$x^2 = (a^2/c^2)y \text{ (parabola)}$$

$$y(c-x) = bc \text{ (hyperbola)}$$

Next development of cubic equation was due to Arabs and Persians. Almahani (c. 860 A.D.) solved the cubic equation of Archimedes but he contributed nothing new. In pursuing the solution of cubic equation , Omar Khayyam (c. 1100 A.D.) blended solid geometry and geometry of conics. He gave geometrical and graphical solutions [3, 4, 7, 8], which are as follows :

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(a) Geometrical solution

let $x^3+a^2x=b$ — (1) be a cubic equation, which is the intersection of two curves

(i) $x^2=ay$ (Parabola)

(ii) $x^2+y^2=(b/a^2)\,x$ (Circle)

Now there are four possibilities :

(1) when a and b are positive particularly we illustrate a=1 and b=2

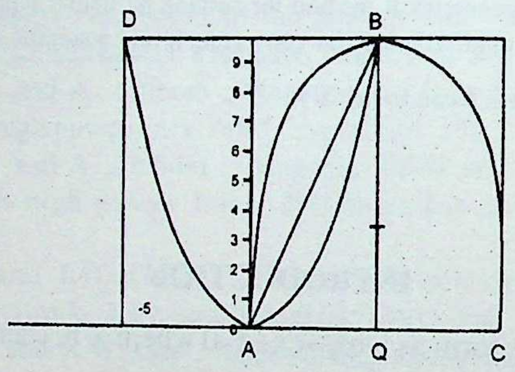

Fig. : 1

AQ=1 is the required solution of eqn (1).

(2) a= -1 (negative) and b=2 (positive), we must use the semicircle in quadrant 4

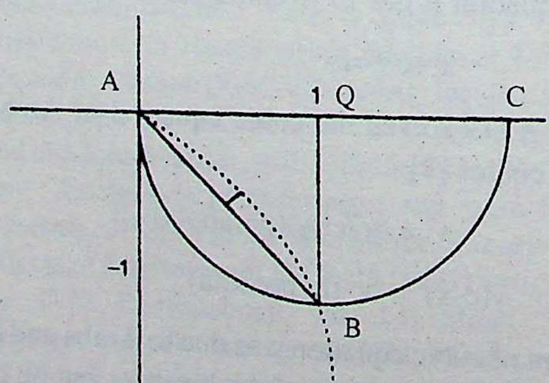

Fig. : 2

AQ =1 is the required solution of eqn (1).

(3) a= -1 (negative) and b= -2 (negative) because diameter b/a^2 can not be negative

so if b is negative, semicircle shifts to the left of the vertical axis.

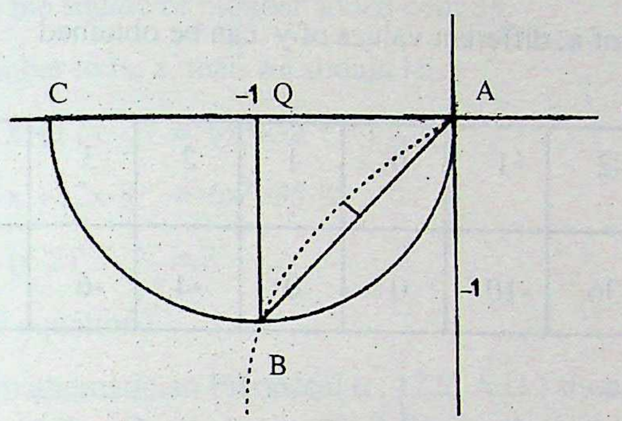

Fig. : 3

AQ=-1 is the solution of eqn(1).

(4) a =1 (positive) and b= -2 (negative)

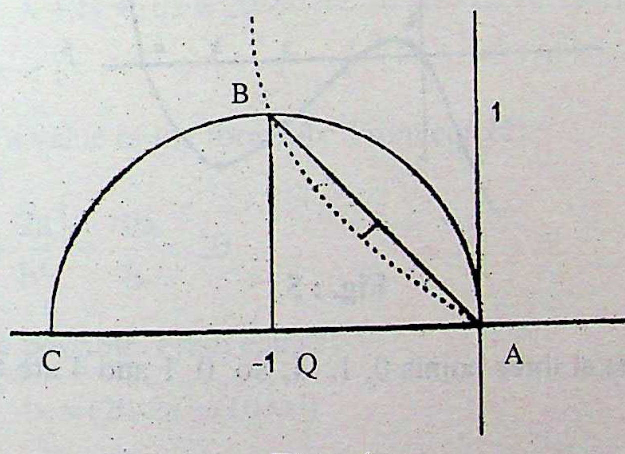

Fig. : 4

AQ=-1 is the solution of eqn(1).

(b) Graphical solution

Graphical solution of cubic equation is known from ancient time. It is also found in Omar Khayyam's work [1,4,6,10]. It is as follows :

Example: let $x^3-5x^2+4x = 0$ be a cubic equation

$$y = x^3 - 5x^2 + 4x$$

for different values of x, different values of y can be obtained

x	-2	-1	0	1	2	3	4	5
y	-36	-10	0	0	-4	-6	0	20

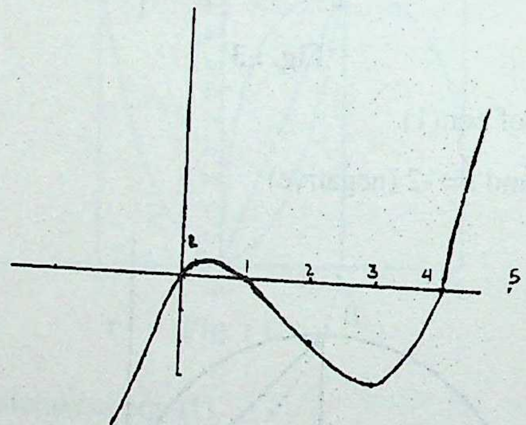

Fig. : 5

Graph cuts the x axis at three points 0, 1, 4, So 0, 1 and 4 are the roots of cubic equation.

Mahāvirāchārya (c 860 A.D.) considered equation of the following form by which only particular type of cubic could be solved [3,7]. It is as follows:

$$Ax^n = q$$

Bhāskarācharyā (c. 1150 A.D.) wrote a problem of cubic equation in his Bijgaṇita [3, 5, 7] which is as follows:

राशिद्वदिशनिघ्नयो राशिधानाड्यच्कः समो यः स्यात्।
राशिकृतिः षड्गुणिताः पञ्चत्रिशद्युता विद्वन् ।। [5, P. 233]

This means

What is the number which multiplied by 12 and increased by the cube of number is equal to six times the square of number added with 35.

Assuming the number to be x, then we should have

$$x^3+12x = 6x^2+35$$

$$x^3+12x-8 = 6x^2+35-8$$

$$(x-2)^3 = 3^3$$

x= 5 is the root of equation.

European mathematician Fibonacci (c. 1225 A.D.) studied cubic equation in his work Felos and Liber Quadratorum [2, 7, 9]. For the problem $x^3+2x^2+10x=20$, he proved that x can not be irrational such as $\sqrt{a+\sqrt{b}}$.

Example: Consider the following cubic equation.

$$x^3+2x^2+10x = 20 \text{ --------------------------------(2)}$$

since $2 < x < 1$

So, x must have a value of the form a/b. From eqn.(2)

$$\frac{a^3}{b^3}+\frac{2a^2}{b^2}+\frac{10a}{b}=20$$

which can not be possible. From eqn.(2)

$$x = (20-2x^2)/(10+x^2)$$

put $x= \sqrt{t}$

$$\sqrt{t} =(20-2t)/ (10+t)$$

which is impossible.

He also gave an approximate solution in sexagesimal form

$$x = 1, 22, 7, 42, 33, 4, 40$$

$$x = 1+\frac{22}{60}+\frac{7}{60^2}+\frac{42}{60^3}+\frac{33}{60^4}+\frac{4}{60^5}+\frac{40}{60^6}$$

In decimal notation

$$x \sim 11368808106$$

Scipio del Ferro (1465 - 1526 A.D.) and Cardan published the general method for solving cubic equation by converting it into quadratic [1, 2, 9]. Let the cubic equation be

$$c_1x^3+c_2x^2+c_3x+c_4 = 0 \quad \text{------------------(3)}$$

or $$x^3+px^2+qx+r = 0 \quad \text{------------------(4)}$$

where $p = c_2/c_1$, $q = c_3/c_1$ $r = c_4/c_1$

put $x = y - p/3$ in eqn. (4)

we get $y^3 + Ay + B = 0$ ------------------(5)

where $$A = \frac{1}{3}(3q - p^2)$$

and $$B = \frac{1}{27}(2p^3-9pq+27r)$$

$$y = m^{1/3} + n^{1/3}$$ be the solution of eqn. (5)

$$y^3 - 3m^{1/3}n^{1/3} - (m+n) = 0 \quad \text{------------------(6)}$$

comparing eqn. (5) and eqn. (6)

$$m^{1/3}n^{1/3} = -A/3$$

$$m + n = -B$$

If m and n are the roots of quadratic equation then equation is-

$$t^2+Bt-A^3/27 = 0$$

$$t = \frac{-B \pm \sqrt{B^2+4A^3/27}}{2} \quad \text{------------------(7)}$$

taking cube root of m and n, we get value of y

$$y = \left[\frac{-B}{2}+\left(\frac{B^2}{4}+\frac{A^3}{27}\right)^{1/2}\right]^{1/3} - \left[\frac{B}{2}+\left(\frac{B^2}{4}+\frac{A^3}{27}\right)^{1/2}\right]^{1/3} \quad \text{------------(8)}$$

three conditions are possible for the roots of equation

1) $B^2+4A^3/27 < 0$, all roots are real but eqn (8) gives them in an imaginary form, thus Cardan's solution is not convenient.

(2) $B^2+4A^3/27 > 0$, one root is real, other two are pair of complex conjugate.

(3) $B^2+4A^3/27 = 0$, cubic consists two equal roots.

Cardan also proved this formula geometrically. The proof of formula is given in the end of paper.

In 1545 A.D. Cardan published the solution of following equation [9]

$$x^3+mx=n \quad \text{--(9)}$$

and $(a-b)^3+3ab(a-b) = a^3-b^3$ --(10)

comparing eqn. (9) and eqn. (10), we get

$$3ab = m$$
$$a^3-b^3 = n$$

then (a-b) is one of the solution of eqn. (9) .

Vieta (b. 1540 A.D.) and Harriat (b. 1560 A.D.) published following method for solving cubic equation (removing x^2 form) [2, 7]

$$x^3 + 3b^2x = 2c^3 \quad \text{------------------------------------(11)}$$

put $x = (e^2-b^2)/e$ in eqn. (11)

we get $e^6-2c^3e^3=b^6$ -- (12)

this equation is quadratic in e^3 and then $x = e-b^2/e$, e, and b^2/e are the roots of expression (11).

Vieta (b. 1540 A.D.) and Girard (b. 1629 A.D.) compared the cubic eqn. (9)

with trigonometrical identity [4, 7]

$$\cos^3\theta - 3/4 \cos\theta = 1/4 \cos 3\theta \qquad \text{------------------------------(13)}$$

Example: Find depth z to which a spherical solid ball made of cork (sp. qr. 1/4) will sink into water

Solution: let radius of ball be a. By Archimedes principle

$$z^3-3az^2+a^3=0 \text{---(14)}$$

put $z = nx+a$ in eqn. (14)

we get $n^3x^3-3a^2nx-a^3= 0$ --(15)

for $x=\cos\theta$ eqn. (13) is identical with eqn. (15)

$$n= 2a \text{ and } \cos 3\theta = 1/2$$

for three distinct values of θ 20^0, 100^0, 140^0 three values of z can be obtained

Required depth $z = a+2a \cos 100^0$

This method is suitable when all roots are real, for imaginary roots identity involving hyperbolic function can be used.

Bomballi (c. 1572 A.D.) introduced complex numbers by which the solution of irreducible case could be found by Cardan rule. It was also given by Kastner (b. 1745 A.D.) and Chariout (b. 1746 A.D.) [2, 7]. Following is an example :

Example: let $x^3-63x-162 = 0$ ---------------------------- (16)

be a cubic eqn. to solve

from eqn. (8)

$$x=[(81+30\sqrt{-3})]^{1/3}-[(81-30\sqrt{-3})]^{1/3}$$

taking cube root $x= -6$ is one of the root of given cubic equation. Dividing the eqn. (16) by (x+6), we get

$$x^2-6x-27= 0$$

where roots are 9 and -3

It means three roots of cubic equation are -6, 9, -3.

In 1668, Hudde gave another method for solving cubic equation [7]

let $x^3 = qx+r$ --(17)

put $x = y+z$ in eqn.(17)

$y^3+z^3 + 3yzx = qx+r$ --(18)

comparing coefficients of like terms

we get $y^3+z^3 = r$

$3yz = q$

solving for y and z we get value of x.

Newton (c.1669A.D.) suggested that a particular root of a equation could be found by substituting a known approximate root in the given equation [4, 7]. Following is an example :

Example:- let the cubic equation be

$y^3 - 2y - 5 = 0$--(19)

for $y = 2$

$y^3 - 2y - 5 = -1$

for $y = 3$

$y^3 - 2y - 5 = 16$

$2 < y < 3$

let $y = 2 + p$ --(20)

from eqn. (19)

$(2 + p)^3 - 2p\ (2+p) - 5 = 0$

$p^3 + 6p^2 + 10p - 1 = 0$--(21)

we get $p = 0.1$ approximately

put $p = 0.1+q$ ---(22)

we have

$$q^3 + 6.3\,q^2 + 11.23q + 0.061 = 0 \text{ ---------------------------------(23)}$$

$q = -0.0054$ approximately

put $q = -0.0054 + r$ ---(24)

from eqn. (23) (omitting r^3 term)

$$6.3r^2 + 11.1619r + .000541708 = 0 \text{ ------------------------------(25)}$$

$r = -0.00004854$ approximately

$r = -0.00004854 + s$----------------------------------(26)

from eqn. (25) (omitting s^2 term because s^2 is very small real number)

$177076885\, s = 1.2584$

$s = 0.00000000\ 7106$ approximately

$s = 0.00000000\ 7106 + t$

being a very small real number we will not consider the value of t

Put the value of s in eqn.(26)

we get

$r = -\ 0.000048532894$

from eqn. (24)

$q = -\ 0.005448532894$

from eqn. (22)

$p = 0.09451467106$

and finally we have

$y = 2.09451467106$ (approximately)

The process terminates if the root is commensurable and it may be carried to any required number of decimal places if it is incommensurable.

De Moivre's (1667-1754 A.D.) method for solving cubic equation was based on the identity [1,4]

$(\text{Cos}\,\theta + i\,\text{Sin}\,\theta)^n = \text{Cos}n\,\theta + i\text{Sin}\,n\,\theta$

let the cubic equation be $ax^3+ bx^2+cx+d =0$ and can be written as

$(x+ b/3a)^3 = (b^3/27)\, a^3 - d/a$ ------------------------------------(27)

if $b^2 = 3ac$

Example : let the cubic equation be

$z^3+6z^2-12z-12 = 0$, from eqn. (27) we get

$(z-2)^3 = 4$

$(z-2)^3 = 4(\cos 2r\pi + \sin 2r\pi)$

$(z-2) = 4^{1/3}(\cos 2r\pi + \sin 2r\pi)^{1/3}$

for r = 0, 1,2 we get three roots. De Moivre's method can be applied when R.H.S. of eqn. (22) is not real. Pure algebric method fails to find the cube root of general complex number in explict form. When $b^2= 3ac$ is not satisfied, this method can be used only particular cases.

Lagrange (1736-1813 A.D.) introduced a method of expressing the root of numerical equation in the form of continued fraction [1].

Example: $x^3-2x-5 = 0$ ------------------------------(28)

root lie between 2 and 3

put $x = 2+1/y$ in eqn. (28)

transformed eqn. is

$y^3-10y^2-6y - 1=0$ ----------------------------------(29)

root lie between 10 and 11

put $y = 10+1/z$ in eqn. (29), we get

$61z^3-94z^2-20z-1 =0$ ------------------------------(30)

$1< z < 2$

put $z = 1 + 1/u$ in eqn. (30)

$54u^3 + 25u^2 - 89u - 61 = 0$

$1 < u < 2$ and so on-----

expression for the root

$$x = 2 + \cfrac{1}{10 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

Geometrical proof of formula

Cardan's geometrical proof of his formula given by eqn. (8) is as follows :

Let the cubic equation be $x^3 + mx = n$-------------------------(31)

where m and n are positive because diamensions of cube be positive. Assuming a cube Z with side t. Where $t = (t-u) + u$

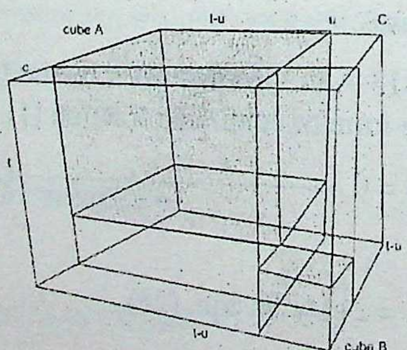

Fig. : 6

This cube of side t contains two cube A and B having sides (t-u) and u and four cuboids.

vol. of cube Z = vol. of cube A+ vol of cube B+vol. of four cuboids

$$t^3 = (t-u)^3 + u^3 + 2tu(t-u) + (t-u)u^2 + u(t-u)^2$$

$$t^3-u^3 = (t-u)^3 + (t-u)\,3tu \quad \text{----------------}(32)$$

let $x = t-u$ then

$$t^3-u^3 = x^3+ 3tux \quad \text{--------------------------}(33)$$

comparing eqn. (31) and eqn. (33)

$$m= 3tu \quad \text{--------------------------------}(34)$$

$$n = t^3- u^3 \quad \text{-------------------------------}(35)$$

from eqn. (34) and eqn. (35), we get

it means $t^6-nt^3-m^3/27= 0$ ----------------------------------(36)

$$t = [+n/2 + (n^2/4 + m^3/27)^{1/2}]^{1/3}$$

from eqn. (35)

$$u = [-n/2 + (n^2/4 + m^3/27)^{1/2}]^{1/3}$$

and finally $x= t-u$ is the solution.

CONCLUSION

Omar Khayyam's, Bhāskarāchāryā's, and Eutocius's methods are suitable only for solving particular type of equations. Methods given by Ferro, Cardan, Bomballi, Vieta, Girard and Newton are general methods. Numerical methods based on convergence of sequences are now used for solving cubics.

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CONSTRUCTION OF SOME IMPORTANT VEDIS FROM ŚULBA SŪTRA

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ABSTRACT

Construction of some vedis from Sulba Sūtra are given.

Key words and phrases: Prācī, Sacifice, Oblong, Citi, Isosceles trapenzium, Altar, Kāmya Agni, Nitya Agni,

Classification Number: 01A32

INTRODUCTION

The science of Mathematics which is said to be the " queen of all sciences" or " the peak of all sciences" has originated in so hoary manner that it is impossiable to trace its origin in any of the written records of the human civilization. Mathematics was known in ancient India as Ganir-veda or the Veda (i.e. Knowledge) of cacluationls for detail description [15].

Only seven Sulba Sūtra are available at present. These are : 1 Baudhāyana, 2. Āpastamba, 3. Vādhula, 4. Maitrāyana, 5. Manava, 6. Vārāha, 7. Kātyāyana. First six belong to Traittriya Samhitā of Krsna Yajurveda. Only the sūtra of Kātyāyana belongs to Śukla Yajur-veda. Out of these Baudhāyana, Āpastamba and Kātyāyana deal with the mathematical portion-namely Geometry, in detial Manava gives a very scanty description of the geometry. It deals only with the Square and Circle. The Baudhāyana, Āpastamba and Manāva make use of these rules in the construction of various citis. But description gives only their dimensions accordingly to their shapes.

The most important points observed in all of them are:-

1. The shape of the citis or the altars of the sacred fires may differ but their areas will remain the same.
2. In the construction of citis, there will be five layers and each layer will consist Department of Mathematics & Statistics, Gurukula Kangri Vishvavidyalaya, Haridwar.of 200 bricks.
3. In the case of the three sacred fires only Garhapatya will have five layers and each layer will consist of 21 bricks [12].

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GLIMPSES OF FIRE-ALTARS

There is fundamental difference between the two types of ritual. The grahya ritual is performed by the householder and his wife in the Gārhapatya fire only. In this ritual, as mentioned in Rv. X 8.5.27 "In this thy (husband's family) affection may increase with offspring, be watehful over the Garhapatya Agni in this house." A priest is seldom required. The srauta ritual is performed by the sacrificer and his wife in more than one fire viz. Gārhapatya, Āhavaniya and Daksina with important role of priest. However there are various views in concern of different agnis on the basis of seasonal and periodical sacrifices as well as Nitya and Kamya.

Kāmya agnis were having five layers, each made of fixed number of bricks. The Nitya fire has 21 brick and kamya fire has 200 bricks in each layer [10].

Name	Horizontal Section/Shape	Sūtra reference
Nitya Agni		
Āhavaniya	Square	Baudhāyana Sulba sūtra (BSS 7.4-7.7)
Gārhaptya	Circle	BSS 7.8
Daksinagni	Semi Circle	[13]

Kamya Agni

Optional fire altars which are built for the fulfilment of desire i.e. Kamyagni Taittariya-Samhita 5.4.11.1-3. These are:- (Each optional fire has an area 7½ square purusa) (see 13).

Name	Horizontal Section/Shape	Sutra reference
Kamya Agni		
Caturasrasyenacit	Hawk bird with square body, wings and tail rectangles	BSS 8.1-8.18 9.2-9.10
Vakrapksavyasta-puccahasyenacit	Hawk bird with bent wings and out spread tail	BSS 10.1-10.14 11.2-11.13

Kankacit	Hawk bird with curved wing and tail	BSS 12.1-12.8
Alajacit	Alaja bird with curved wings and tail	BSS 13.1-13.5
Praugacit	Isosceles triangle	BSS 14.4-14.8
Ubhayata praugacit	Rhombus	BSS 15.2-15.6
Rathacakracit	Chariotwheel	BSS 16.3-16.5, 16.6-16.20
Dronacit	Square trough	BSS 17.1-117.12
	Circular trough	BSS 18.1-18.15
Samasanacit	Isosceles trapezium	BSS 19.1-19.11
Kurmacit	Tortoise	BSS 20.1-20.21 21.2-21.13
Vedi		
Māhavedi/ Saumikyāvedi	Isocels trapezium (face a =, base b = 30, height c = 36)	BSS 4.3 [2] & ([3], p.69) ([7], p.22)
Sautramanivedi	Isosceles trapezium $(a = 8\sqrt{3},\ b = 10\sqrt{3},\ c = 12\sqrt{3})$	BSS 3.12, 1.85
Paitrkivedi	Isoceles trapezium (a = 8, b = 10, c = 12)	BSS 1.82, 3.11 ([17], p. 68)
	Square having four corners in four corrdinate directions.	
Ultaravedi	Four cornered and each side is 36 angulas	BSS 3.10
Asvamedhavedi	Isosceles trapezium (a = 32, b = 40, c = 54)	

(Units are in padas otherwise stated)

Following are some of the geometrical methods which are used for construction of altars.

1. Construction of a trapeziam Similar to a given trapezium.
2. Construction of trapezium similar to double of its area.
3. Construction of a trapezium similar to third part of its area.
4. Construction of an isosceles trapezium similar to given trapezium by increas ing/diminishing its sides by a certain proportion.
5. Construction of an isosceles trapezium of given area.
6. Construction of a square of given area
7. A quadrilateral formed by the lines joining the middle parts of the sides of a square is a square whose area is half that of the original one.
8. A quadrilateral formed by the lines joining the middle points of the sides of the rectangle is a rhombus whose area is half that of the rectangle.

In addition ot geometical problems. Sulb Sutras also concern extensively with the use of instruments such as bomboo rod, geometrical compass, peg, rope and sanku for altar construction; relative distance between Āhavaniya, Daksināgni, Gārhapatya and Uttaravedi; relative position of various altars with regard to Māhavedi; enlargement of fire altars; a few indeterminate problems; a little of geometrical algebra; fractions; elementry concept of arithmetic series [13].

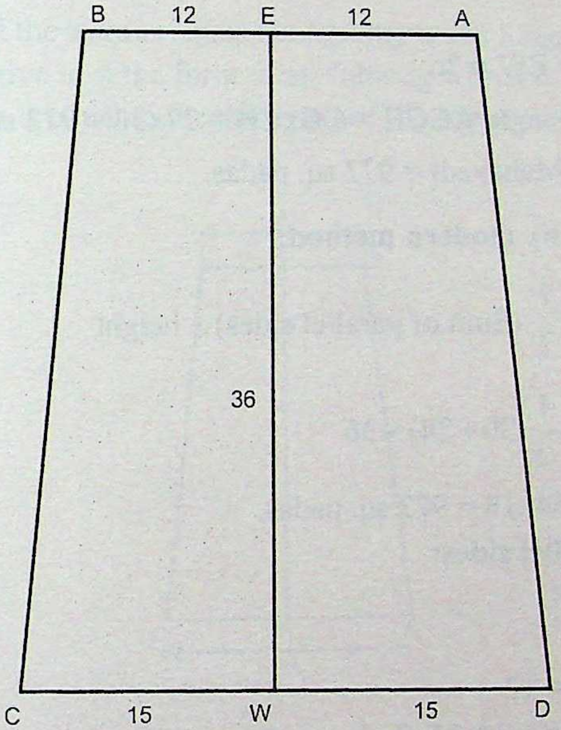
MAHĀVEDI

त्रिंश॑त्पदानि प्रक्रमा वा पश्चात्तिरश्ची भवति षट्त्रिंश॑त्प्राची चतुर्विंश॑शतिः पुरस्तात्तिरश्चीति महावेदेर्विज्ञयाते मानयोगस्तस्या व्याख्यातः।। (BSS 1.90)

The meaning of above Sūtra is as follows:

"The western side is thirty padas of prakrams long, the prācī thirty-six the eastern side twenty four" this is the tradition for Mahāvedi (the vedi used at the soma sacrifices); it has been explained how it is to be measuued out [17].

In other words above may be stated as Mahāvedi described as isosceles trapezium measuring 80 padas or prakarmas on western side, 36 padas along east-west line 24 padas on eastern side [13].



4 padas = 1 cm

(see 13)

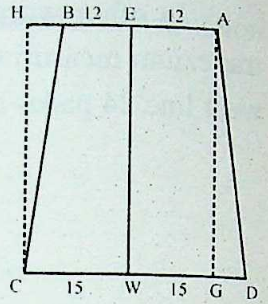
Description :

Mahāvedi ABCD is isosceles trapezium of width 36 and parallel sides 24 and 30 units. This is constructed by first making the central-axis (east-west line), points W and E and then drawing the four segments WD, WC, EA and EB along the other cardinal directions (south and north) ([13], p.22) Mahavedi is called Saumiki vedi in ASS.

Calculation of area of Mahavedi by Śulba Sūtra method:

Draw a perpendicular form A on side CD and from C on AB. Let perpendiculars meet on WD at G and on AB at H.

Area of Mahavedi ABCD = Area of rectangle AGCH.



E and W are middle points of sides AB and CD.

$\therefore$ BE = EA = 12

$\therefore$ CW = WD = 15

$\therefore$ WG = 12 and CG = 15+12 = 27

$\because$ EW = 36

$\therefore$ AG = HC = EW = 36

$\therefore$ Area of rectangle AGCH = CGxCH = 27x36 = 972 sq. padas [4].

Hence area of Mahavedi = 972 sq. padas.

Verificaton of area by modern method:

Area of Mahavedi $= \frac{1}{2}$ (Sum of parallel sides) x height

$$= \frac{1}{2}(30+24)\times 36$$

= 54x18 = 972 sq. padas.

Length of non-parallel sides:

Since EW = AG

$\therefore$ AG = 36

and EA = WG = 12

$\therefore$ GD= WD-WG=15-12=3

Now in triangle AGD, using Sulbkara's theorem

$AD^2=AG^2+GD^2$

$\therefore AD^2 = (36)^2+ (3)^2$

$= 1296+9=1305$

$\therefore AD=\sqrt{1305} =36.12$

$\therefore$ AD=BC=36.12 padas.

SAUTRĀMANIVEDI

वेदितृतीये यजेतेति सौत्रामणिकी वेदिमभ्युपदिशन्ति॥ (BSS 1,85)

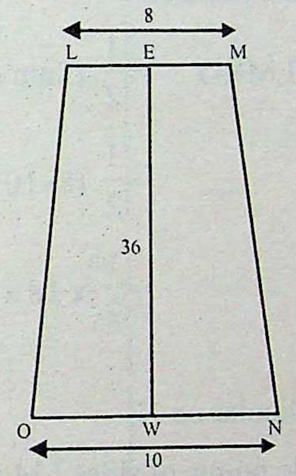
"He is to sacrifice on the third part of the vedi" thus it explains the formation of vedi of the Sautramani sacrifice.

महावेदेस्तृतीयेन समचतुरस्रकृताया अष्टादशपदा पार्श्वमानी भवति॥ (BSS 1.86)

The side of that area which is formed into a square with the third part of the Mahavedi is eighteen padas long.

तस्यै दीर्घकरण्यामेकतोऽणिमत्करण्यां च याथाकामीति ॥ (BSS 1.87)

The meaning of the sutra is : instead of giving to the Sautrāmanivedi the form of a square we may give to it the form of an "oblong which is shorter on one side" and the area of which is equal to the third part of the Mahavedi [17]..



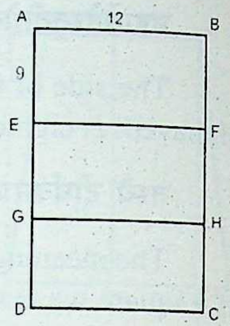
Description :

The Sautrāmanivedi is constructed exactly in the same manner as the Mahavedi. Measures prescribed for the Mahavedi are 36 padas for prāci (east-west line), 30 padas for western side, 24 padas for eastern side all these lengths were devided by three then we obtain 12 padas for prācī, 10 padas for western side and 8 padas for estern side. The area of figure formed is one ninth of the area of Mahavedi which is 108 units of area; Multiplying it by 3, we get 324 units of area which is equal to the area of Sautramanivedi and dimensions of Sautramani vedi are as follows 8,10 and 36 padas [4].

Measurment of area of Sautramanivedi by Sulba Sutra method:

AB = 12 Padas

AE = EG = GD = 9 padas



$\therefore$ AF = FG = HD = 108 sq. padas

$\therefore$ AC = 324 sq. padas.

Area of rectangle ABFE = 12X9 = 108 sq. padas

$\therefore$ Area of rectangle ABCD = 12X27 = 324 sq. padas

Area of Sautramanivedi LMNO = Area of rectangle ABCD = 324 sq. padas [4].

Varification of area by modern method :

Area of Sautramanivedi LMNO $= \frac{1}{2}$ (Sum of parallel sides) x height

$= \frac{1}{2}$ (8+10) x 36

$= \frac{1}{2}$ x 18 x 36 = 9 x 36 = 324 sq. padas

Length of non-parallel sides:

From fig., MP is perpendicular on N.

Since E and W are middle points of sides LM and ON

$\therefore$ LE = EM = 4 and OW=WN=5

$\therefore$ PN = WN-WP

= 5-4 = 1 ($\because$ WP = EM)

Since EW = 36

$\therefore$ MP = 36

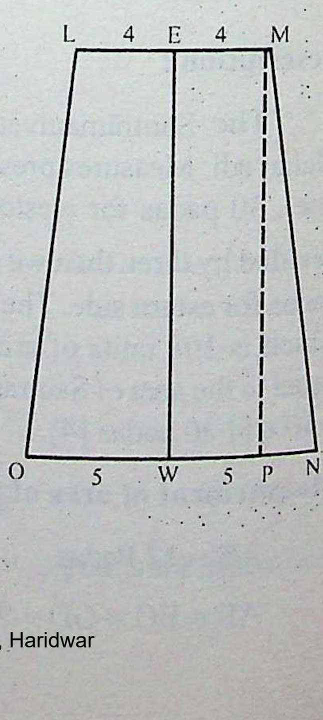
In triangle MPN, using Sulbkara's theorem

$MN^2 = MP^2 \quad PN^2$

$= (36)^2 + (1)^2$

$\therefore$ MN = $\sqrt{1297}$ =36.01 padas

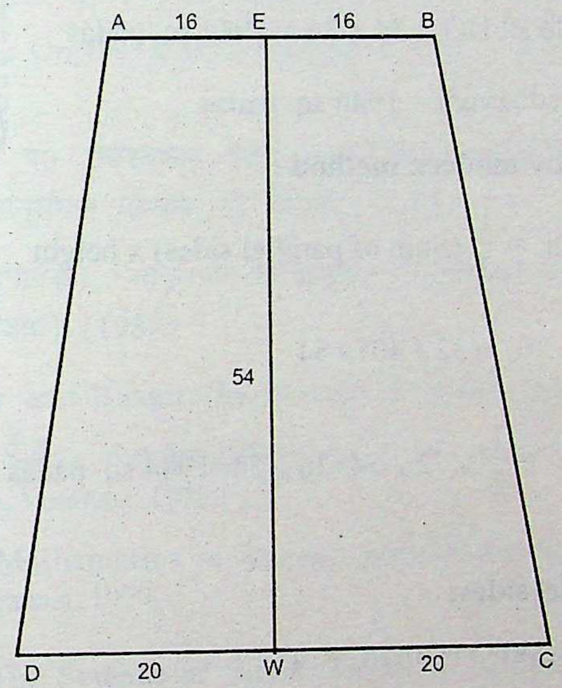
$\therefore$ MN = LO=36.01 padas



AŚVAMEDHA VEDI

द्विस्तावा वेदिर्भवतीत्यश्वमेधे विज्ञायते॥ (ASS 5.36)

Area of the altar for Aśvamedha is double the area of Saumikivedi (Mahavedi) i.e. 1944 sp. pada [4].



Description :

Because area of Mahavedi is 972 sq. padas therefore area of Aśvamedhavedi will be 1944 sq. padas. Shape of this vedi is the same as that of Mahavedi. Accordingly prācī of Aśvamedha vedi is 54, western side is 40 and eastern side is 32 padas [4].

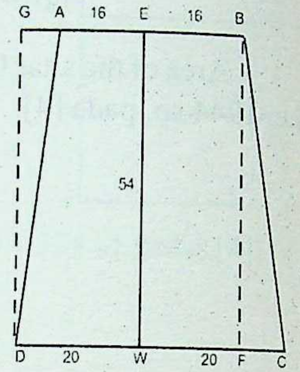
Now in figure ABCD is Aśvamedhavedi.

Measurement of area of Asvamedhavedi by Sulba Sutra Method :

Draw two perpendicular one from B on DC and second from D on BA. Let perpendiculars meet on WC at F and on BA at G.

Area of rectangle BFDG= Area of Aśvamedhavedi ABCD

Since E and W are middle points of the sides AB and DC.



$\therefore$ AE=EB=16 and DW=WC=20

also GD=EW=BF=54

$\therefore$ DF=DW + WF = 20 + 16 = 36

$\because$ Area of rectangle BFDG = 54 x 36 = 1944 sq. padas

Hence area of Aśvamedhavedi = 1944 sq. padas.

Varification of area by modern method :

Area of Asvamedhavdi $= \frac{1}{2}$ (Sum of parallel sides) x height

$$= \frac{1}{2}(32 + 40) \times 54$$

$$= \frac{1}{2} \times 72 \times 54 = 36 \times 54 = 1944 \text{ sq. padas}$$

Length of non-parallel sides:

from fig., FC = WC - WF = 20-16 = 4

and BF = 54

In triangle BFC, using Sulbkara's theorm

$BC^2 = NF^2 + FC^2$

$BC^2 = (54)^2 + (4)^2$

$= 2916 + 16 = 2932$

$\therefore$ BC = $\sqrt{2932}$ = 54.15

$\therefore$ BC = AD = 54.15 padas.

In this paper the dimensions of Mahavedi, Sautramanivedi and Aśvamedhavedi and their shapes are described. Their areas have been calculated by Śulba Sūtra method and modern method. In each case area comes out to be equal, showing the exactness of the geometrical methods described in Śulba Sūtra.

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KINETICS AND MECHANISM OF MN (II) CATALYSED PERIODATE OXIDATION OF O-CHLORO ANILINE IN AETONE-WATER MEDIUM

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ABSTRACT

Kinetic-mechanistic studies made for periodate oxidation of o-chloroaniline (OCA) in acetone-water medium, have been used for derivation of rate law and proposing mechanism which satisfy various observations like first order in each reactant and catalyst, stoichiometry (1 mol OCA : 2 mol periodate), kinetic parameters (Ea = 6.5 K cals/mol; A = 1.45 $\times 10^5$ lit/mol/sec; $\Delta S^{\neq}$ = –38.0 EU; $\Delta F^{\neq}$ =17.72 K Cals/mol; $\Delta H^{\neq}$ = 5.9 K Cals/mol), main product identified (2-chloro-1,4 benzoquinone), and the effect of pH, dielectric constant, and ionic strength on reaction rate.

Key words : Kinetics and mechanism, Mn (II) catalyst, Periodate oxidation, o-chloroaniline, 2-chloro-1,4-benzoquinone.

INTRODUCTION

Reports on kinetic-mechanistic studies on the Mn (II) catalysed periodate oxidation of aromatic amines are very few[1-4]. In continuation to our earlier communications on uncatalysed oxidations of such type [5-7], present paper deals with the kinetic studies made on Mn (II) catalysed periodate oxidation of OCA in acetone-water medium.

MATERIALS AND METHODS

Chemicals of E.Merck/CDH A.R. grade were used after distillation/recrystallization. Triply distilled water was used for preparation of the solutions. The progress of the reaction was followed spectrophotometrically [5] by recording the absorbance at 475 nm, i.e. the λ_{max} of reaction mixture in the duration in which the λ_{max} did not change. The pH was maintained at 5.0 by using Thiel, Schultz and Koch buffer [6,7]

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in all kinetic runs except when the effect of pH was studied, Plane mirror method and Guggenheim's method were used for evaluation of initial rates $[(dA/dt)_i]$ and pseudo first order rate constant k_1 or second order rate constant k_2 respectively. NaCl solutions were used for maintaining the ionic strength (μ) in the kinetic runs.

RESULTS AND DISCUSSION

1 mol OCA consumed 2 moles of periodate as determined iodometrically. The data (table-1) indicated 2nd order for the reaction, being first order in each reactant. Linear relation between concentration of the reactants and rate supported the 2nd order kinetics. In pseudo first order conditions (table-2), the $[(dA/dt)]^{-1}$ or k_1^{-1} vs$[S]^{-1}$ Plots were linear with almost negligible intercept, suggesting the Michaelis-Menten type kinetics being followed with respect to both reactants with the possiblity of formation of a fast decaying intermediate complex between reactants [10,11]. Data in table-3 established the 1st order in catalyst. Rate-pH profile showed a maxima at pH=5.0 (table-4). A linear relation between log (dA/dt)i or log k_2 and 1/D with negative slope (where D is the dielectric constant of the medium) and a primary linear type plot between log $(dA/dt)_i$ or log k_2 vs ionic strength (μ) that were obtained from the data in table-5, indicated an ion-dipole interaction in this reaction. Arrhenius plot was made between $30 \pm 0.1°C$ to $45 \pm 0.1°C$ and the values of different thermodynamic parameters evaluated taking $[OCA]=1.0 \times 10^{-3}M$, $[NaIO_4]=1.0 \times 10^{-4}M$, $[Mn^{++}]=8.0 \times 10^{-6}$ M and acetone = 10.0% (v/v), are Ea=6.5 K cals/mol; A=1.45×10^5 lit/mol/sec ; $\Delta S^{\neq}$=-38.0 EU ; $\Delta F^{\neq}$=17.72 K cals/mol ; $\Delta H^{\neq}$=5.9 K cals/mol. Low value of energy of activation and high frequency factor are characteristic of a bimolecular reaction in the solution in the which the reacting species are larger in size. A large negative value of $\Delta S^{\neq}$ suggests the formation of strongly solvated, charged and rigid transition state.

Table-1. Determination of order w.r.t. reactants.

λ_{max} = 475 nm; pH= 5.0; Acetone = 10.0% (v/v); Temp = 30 ± 0.1°C;

$[Mn^{++}]=8.0 \times 10^{-6}$M ($6\times10^{-6}$ M for variation of $[NaIO_4]$)

$[OCA] \times 10^4$M	2.0	4.0	6.0	8.0	10.0	10.0	10.0	10.0	10.0	10.0	10.0
$[NaIO_4]\times10^3$M	1.0	1.0	1.0	1.0	1.0	0.1	0.2	0.3	0.4	0.8	1.0
$(dA/dt)\times10^2(min^{-1})$	3.0	5.6	8.0	10.3	12.4	7.0	14.0	27.0	40.0	54.0	64.0

Table-2. λ_{max} =475 nm; pH=5.0; Acetone = 10.0%(v/v); Temp = 30.0 ± 0.1°C;

$[Mn^{++}]=8.0\times10^{-6}$ M (6×10^{-6} M for variation of $[NaIO_4]$).

$[OCA]\times10^3$M	1.0	1.2	1.4	1.6	1.8	0.1	0.1	0.1	0.1	0.1	0.1
$[NaIO_4]\times10^3$M	0.10	0.10	0.10	0.10	0.10	1.0	2.0	3.0	4.0	5.0	6.0
$(dA/dt)_i\times10^3(min^{-1})$	6.5	14.0	21.0	27.5	32.5	7.0	15.0	22.0	26.0	34.0	40.0
$k_1\times10^3$(sec-1)	1.73	2.01	2.30	2.50	2.78	1.54	2.30	3.07	3.45	3.84	4.22

Table-3. Determination of order w.r.t. Mn^{++}

$[NaIO_4]=1.0\times10^{-4}$M;$[OCA]=1.0\times10^{-3}$M;Temp. = 30.0 ± 0.1°C;

Acetone =5.0% (v/v); λ_{max} =475 nm

$[Mn^{++}]\times10^{-6}$M	4.0	6.0	8.0	10.0	12.0	14.0
$(dA/dt)_i\times10^3(min^{-1})$	6.5	10.0	13.0	16.0	19.0	22.0

Table-4. Rate-pH profile.

$[OCA]=1.0\times10^{-3}$ M; $[NaIO_4]$ = 1.0×10^{-4} M; λ_{max} =475 nm; Acetone = 5.0% (v/v);

Temp. = 30.0 ± 0.1°C; $[Mn^{++}] = 8.0\times10^{-6}$M

pH	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5
$(dA/dt)_i\times10^3(min^{-1})$	5.0	5.25	6.5	8.0	5.5	5.0	4.5	4.0	3.0

Table-5 Effect of D and μ on reaction rate.

[OCA]= 1.0×10^{-3} M; [$NaIO_4$] =1.0×10^{-4} M; λ max= 475 nm; * Acetone= 5.0% (v/v); Temp.= 30.0 ± 0.1°C; [Mn^{++}]= 8.0 x 10^{-6} M.

D	72.4	71.0	69.7	68.4	--	--	--	--
* μ x 10^2	--	--	--	--	10.0	12.3	14.1	17.3
$(dA/dt)_i$ 10^3 (min^{-1})	9.0	8.0	7.0	6.0	6.5	7.0	7.5	8.25
k_1 x 10^3(Sec^{-1})	3.65	2.40	1.73	1.34	1.54	1.60	1.63	1.73
k_2 (1. mol^{-1}.(Sec^{-1})	3.65	2.40	1.73	1.34	1.54	1.60	1.63	1.73

The increase in the rate from pH 3.5 to 5.0 may be due to the decrease in the protonation of OCA from pH 3.5 to 5.0 (table-4) which makes greater concentration of OCA available for the reaction. This leads to the assumption that unprotonated OCA is the reactive species in present case. Second part of this profile suggests that the periodate monoanion [$I\bar{O}_4$] is the reactive species of periodate [9] whose concentration goes on decreasing with increase in pH beyond the value 5.0 [9] decreasing thereby the rate of reaction beyond pH 5.0.

Based on these results, the mechanism may be proposed (Chart) involving the lone pair of electrons on nitrogen atom of OCA for the co-ordinate bond formation between OCA and Mn^{++} species in a reversible step to from complex C_1 [step (i)] which, in turn, interacts with $I\bar{O}_4$ in slow and rate determining step (ii) to give C_2. C_2 Changes by fast hydrolysis in to C_3. The formation of a charged intermediate complex C_2 by the attack of $I\bar{O}_4$ on the nitrogen of anilino group and stabilization of positive charge on nitrogen of this group, have also been observed and supported by LFER studies for the uncatalysed periodate oxidation of few aromatic amines [8] In addition, a high negative value of entropy of activation and the effect of dielectric constant on the reaction rate support the involvement of solvation effects in this reaction.

It should also be noted that the initial part of the reaction is significant in the present case and the second molecule of $I\bar{O}_4$ reacting later to give C_4 is not significant. C_4 changes by fast hydrolysis to give C_5 i.e. the main product of raction that has been isolated and seperated by employing the method reported eariler in case of other aromatic amines [6,7]. It was characterized as 2-chloro-1,4- benzoquinone on the basis of test for quinone [4], M.P. [12], UV-VIS, IR and H-nmr spectra (showing

characteristic bands for C-C1 [13] and benzoquinones [14]). The overall process may be representead as follows:

$$OCA + Mn^{++} \underset{k_2}{\overset{k_1}{\rightleftarrows}} C_1 \text{---------------------(i)}$$

$$C_1 + [IO_4] \underset{slow}{\overset{k_3}{\rightarrow}} C_2 + Mn^{++} \text{---------------------(ii)}$$

$$C_2 \underset{H^+, H_2O}{\overset{Fast}{\rightarrow}} C_3 + H_2O + HIO_3 \text{---------------------(iii)}$$

$$C_3 + IO_4^- \xrightarrow{+H^+} C_4 + H_2O + HIO_3 \text{-----------------------------------(iv)}$$

$$C_4 + \underset{Fast}{\overset{+H_2O}{\rightarrow}} C_5(2-\text{chloro}-1,4-\text{benzoquinone}) \text{-------------(v)}$$

On applying steady state treatment to C_1, the rate law in terms of rate of loss of $[IO_4]$ may be derived as follows:

Rate of loss of $[I\bar{O}_4]$ or-d $[IO_4^-]/dt = k_3 [C_1] [IO_4^-]$

= Rate of loss of C_1 or- d $[C_1]/dt$ -------------(i)

Rate of formation of $C_1 = +d[C_1]/dt = k_1[OCA][Mn^{++}] - k_2 [C_1]$

At steady state,- d $[C_1]/dt = +\, d[C_1]/dt$

Therefore, $k_3 [C_1] [IO_4^-] = k_1 [OCA] [Mn^{++}] = k_2 [C_1]$

or $[C_1] \{k_3. [IO_4^-] + k_2\} = k_1 [OCA] [Mn^{++}]$

$$\text{or } [C_1] = \frac{k_1[OCA][Mn^{++}]}{k_3[IO_4^-] + k_2} \text{----------------------(2)}$$

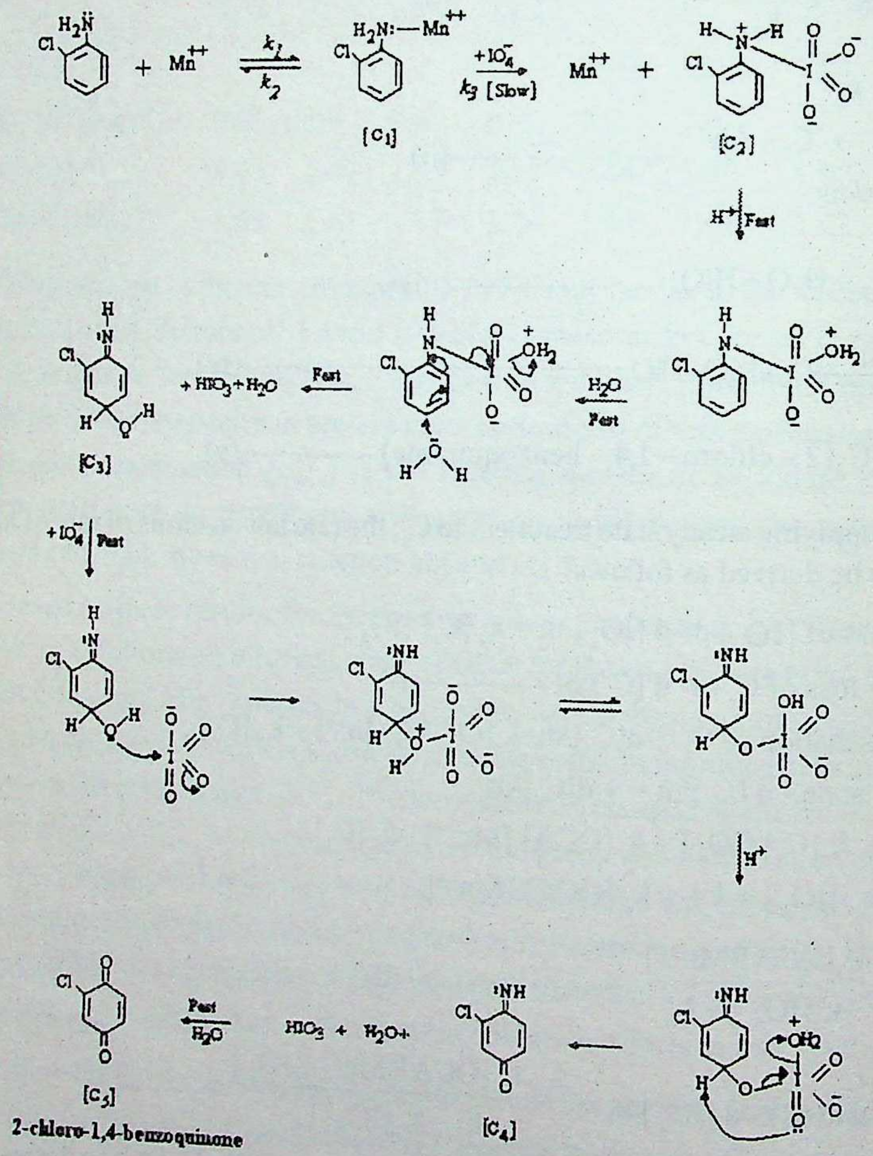
$$\text{From (1) and (2),- d } [IO_4^-]/dt = \frac{k_3.k_1.[OCA][Mn^{++}][IO_4^-]}{k_2 + k_3[IO_4^-]} \text{----------(3)}$$

Since step (ii) is slow and rate determining step, hence $k_3 [IO_4^-] << k_2$ may be assumed. Therofore, the rate law (3), may be written as rate law (4) which explains all the kinetic results.

$-d[IO_4^-]/dt = k_{obs}.[OCA][Mn^{++}] [IO_4^-]$-------------------(4)

where, $k_{obs} = k_3.\, k_1 / k_2$

CHART



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STRONG SUMMABILITY OF FUNCTIONS BASED ON (D,α,β) (C,ℓ,m) SUMMABILITY METHODS

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(Received 10.08 2004)

ABSTRACT

Singh [7] defined strong summability methods. Here, we have defined summability of function based on (D,α,β), (C,ℓ,m) summability methods and investigated some of their properties.

Key words and Pharase: Strong summability, function, (D,α,β), (C,ℓ,m) summability methods, Lebesgue integral, Holder's inequality and Minkowski's inequality.

Mathematics subject classifications (2000) : 40F05.

SOME NOTATIONS AND DEFINITIONS

Let $s(x)$ be any function which is Lebesgue integrable in $(0,x)$ for all finite $x > 0$ and that $s(x)$ is bounded in some right hand neighbourhood of the orgin. Integrals of the form $\int_0^\infty$ are throughout to be taken as $\lim_{x\to\infty}\int_0^x$, $\int_0^x$ being taken as a Lebesgue integral. We write

$$\partial_{\ell,m}(X) = \begin{cases} S(x) & l=0 \\ \left\{\dfrac{\Gamma(\ell+m+1)}{\Gamma(\ell)\Gamma(m+1)}\right\}\dfrac{1}{X^{\ell+m}}\displaystyle\int_0^x (x-y)^{\ell-1}y^m s(y)dy, & \ell>0, m>-1 \end{cases} \tag{1.1}$$

We also write

$$g^{\alpha,\beta}(y) = \left\{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\right\}y^{\beta+1}\int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}s(x)dx, \tag{1.2}$$

if this exists. We also write

$$\mathrm{U}_{\alpha,\beta,\ell,m}(\mathrm{y}) = \left\{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\right\}\mathrm{y}^{\beta+1}\int_0^\infty \frac{\mathrm{x}^{\alpha-1}}{(\mathrm{x}+\mathrm{y})^{\alpha+\beta+1}}\partial_{\ell,m}(\mathrm{x})d\mathrm{x}, \tag{1.3}$$

if this exists, where $\ell>0, m>-1, y>0, \beta>-1, \alpha>0$.

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With the usual terminology, we say that $s(x)$ is summable

(i) (C,ℓ,m) to the sum s, if $\partial_{\ell,m(x)}(x)$ exists, and tends to a limit s as $x \to \infty$,

(ii) $[C,\ell,m]_p$ or strongly summable (C,ℓ,m) with index p to the sum s, where $\ell \geq 1, m \geq -1, p \geq 1$, if

$$\int_0^x | \partial_{\ell-1,m}(y) - s|^p dy = o(x) \text{ as } x \to \infty \tag{1.4}$$

(iii) (D,α,β) to the sum s, it $g^{\alpha,\beta}(y)$ tends to s as $y \to \infty$,

(iv) (D,α,β) (C,ℓ,m) to the sum s, if $U_{\alpha,\beta,\ell,m}$ (y) tends to s as $y \to \infty$.

Definition (ii) & (iii) are due respectively to Mishra and Srivastava [5] and Kwee [3], while (iv) is due to Mishra and Singh [6].

Let

$$s(x) = \int_0^x a(y)dy, \tag{1.5}$$

where a (y) is Lebesgue integrable in $(0, x)$ for any finite $x > 0$. Since $U_{\alpha,\beta,\ell-1,m}(y)$ converges for all $y > 0$, then it is easy to justify the differentiation under integral sign $(\int)$ and that

$$y\frac{d}{dy}U_{\alpha,\beta,\ell,m}(y) = -(\ell+m)[U_{\alpha,\beta,\ell,m}(y) - U_{\alpha,\beta,\ell-1,m}(y)] \tag{1.6}$$

$$= B_{\alpha,\beta,\ell,m(y)} \qquad (\ell \geq 1),$$

where

$$B_{\alpha,\beta,\ell,m}(y) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1} \int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}} \tau_{\ell,m}(y)dy.$$

Also we know that

$$y\frac{d}{dy}\partial_{\ell,m}(y) = (\ell+m)[\partial_{\ell-1,m}(y) - \partial_{\ell,m}(y)] = \tau_{\ell,m}(y) \tag{1.7}$$

$$= \frac{\Gamma(\ell+m+1)}{\Gamma(\ell)\Gamma(m+1)}\frac{1}{y^{\ell+m}}\int_0^y (y-v)^{\alpha-1}v^m va(v)dv.$$

Before giving the justification of (1.6) we first make some remarks on (1.7). If we do not assume (1.5), then $\tau_{\ell,m}(y)$ has not been defined, so that the second equation (1.7) is meaningless. However the first equality holds every where if $\ell \geq 2$ and almost every where if $2 > \ell \geq, 1$. if we assume (1.5), then both the equations in (1.7) hold every where for $\ell \geq 1$. $y \geq 1$, the middle term in (1.7) is not defined. However, the equality

$$y\frac{d}{dy}\partial_{\ell,m}(y) = \tau_{\ell,m}(y)$$

continues to hold almost every where for $1 > \ell > 0$.

In order to get a valid proof of (1.6), under the assumption that the integral defining $U_{\alpha,\beta,\ell-l,m}(y)$ converges for all $y > 0$, that it is necessary and sufficient for the convergence of (1.3), that

$$\int_1^\infty \frac{\partial_{\ell-1,m}(x)}{x^{\beta+2}}dx \text{ should converge} \tag{1.8}$$

We write

$$R_{\ell-1,m}(y) = \int_y^\infty \frac{\partial_{\ell-1,m}(x)}{x^{\beta+1}}dx$$

which is bounded for $y \geq 0$ and tends to zero as $y \to \infty$.

Now

$$\partial_{\ell,m}(u) = \frac{\ell+m}{u^{\ell+m}}\int_0^u v^{\ell+m-1}\ \partial_{\ell-1,m}(v)dv$$

$$= -\frac{(\ell+m)}{u^{\ell+m}}\int_0^u v^{\ell+m-1}v^{\beta+R}dR_{\ell-1,m}(v)$$

$$= -\frac{(\ell+m)}{u^{\ell+m}}\int_0^u v^{\ell+m+\beta+1}dR_{\ell-1,m}(v) \tag{1.9}$$

Since $R_{\ell-1,m}(v)\to\infty$, therefore given $\in >0$, there is number u_0 such that $|R_{\ell-1,m}(v)|$ $<\varepsilon$ for $v>u_0$, we also note that $|R_{\ell-1,m}(v)| = O(1)$ for $v<u_0$. Hence we see from (1.9), on integration by parts, that

$$\partial_{\ell,m}(u) = -\frac{\ell+m}{u^{\ell+m}}\{v^{\ell+m+\beta+1}R_{\ell-1,m}(v)\} + \frac{\ell+m}{u^{\ell+m}}\int_o^u \frac{d}{dv}(v^{\ell+m+\beta+1})R_{\ell-1,m}(v)dv$$

$$= -\frac{\ell+m}{u^{\ell+m}}\int_o^{u'}\frac{d}{dv}(v^{\ell+m+\beta+1})R_{\ell-1,m}(v)dv + \frac{\ell+m}{u^{\ell+m}}\int_{u'}^{u}\frac{d}{dv}(v^{\ell+m+\beta+1})R_{\ell-1,m}(v)dv$$

$$= O(u^{\beta+1}) + o(u^{\beta+1})$$

$$= o(u^{\beta+1}), \text{ if } u\to\infty$$

Hence $\partial_{\ell,m}(u) = O(u^{\beta+1})$, as $u\to\infty$ (1.10)

Let $H_{\alpha,\beta,\ell,m}(y) = \int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\partial_{\ell,m}(x)dx$

$$H_{\alpha,\beta,\ell,m}(\eta+y) = \int_0^\infty \frac{x^{\alpha-1}}{(x+\eta+y)^{\alpha+\beta+1}}\partial_{\ell,m}(x)dx$$

Let $y>y_0$ to be fixed. By Lagrange's for mula, for the remainder in Taylor's series, we have, for $\eta>-y$, and for all $x\geq 0$

$$\frac{1}{\eta}\{H_{\alpha,\beta,\ell,m}(\eta+y) - H_{\alpha,\beta,\ell,m}(y)\} = \frac{1}{\eta}\int_0^\infty x^{\alpha-1}\partial_{\ell,m}(x)\left\{\frac{1}{(x+n+y)^{\alpha+\beta+1}} - \frac{1}{(x+y)^{\alpha+\beta+1}}\right\}dx$$

By Lagrange's formula, we have

$$\frac{1}{\eta}\left\{\frac{1}{(x+\eta+y)^{\alpha+\beta+1}}-\frac{1}{(x+y)^{\alpha+\beta+1}}\right\}=-\frac{(\alpha+\beta+1)}{(x+y)^{\alpha+\beta+2}}+\frac{\eta}{2}\frac{(\alpha+\beta+1)(\alpha+\beta+2)}{(\xi+y)^{\alpha+\beta+3}}\ .$$

for some ξ between $(x+y)$ and $(x+\eta+y)$ (hence ξ depends on x,y,η). Now choose a fixed y' with $0<y'<y$. Then, for $\eta>-(y-y')$ (and thus, in particular, for all sufficiently small η), we have $\xi \geq y$,

$$\left|\frac{1}{\eta}\{H_{\alpha,\beta,\ell,m}(\eta+y)-H_{\alpha,\beta,\ell,m}(y)\}+\int_0^\infty \frac{(\alpha+\beta+1)x^{\alpha-1}}{(x+y)^{\alpha+\beta+2}}\partial_{\ell,m}(x)dx\ \right|$$

$$\leq\frac{|\eta|}{2}\ \left|\int_0^\infty x^{\alpha-1}\partial_{\ell,m}(x)\frac{(\alpha+\beta+1)(\alpha+\beta+2)}{(x+y)^{\alpha+\beta+3}}dx\ \right|\ .$$

$$\leq o\,(|\eta|).$$

Since the integral on the right hand side converges. Hence $H_{\alpha,\beta,\ell,m}(y)$ is differentiable, and

$$\frac{d}{dy}H_{\alpha,\beta,\ell,m}(y)=\lim_{\eta\to o}\frac{1}{\eta}[H_{\alpha,\beta,\ell,m}(\eta+y)-H_{\alpha,\beta,\ell,m}(y)]$$

$$=-(\alpha+\beta+1)\int_0^\infty\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+2}}\partial_{\ell,m}(x)dx. \qquad (1.12)$$

We know that

$$U_{\alpha,\beta,\ell,m}(y)=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^\infty\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\partial_{\ell,m}(x)dx$$

Differentiating this equation, we have

$$\frac{d}{dy}U_{\alpha,\beta,\ell,m}(y)=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\Big[(\beta+1)y^{\beta}\ \int_0^\infty\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\partial_{\ell,m}(x)dx$$

$$-\ y^{\beta+1}\int_0^\infty\frac{x^{\alpha-1}(\alpha+\beta+1)}{(x+y)^{\alpha+\beta+2}}\partial_{\ell,m}(x)dx\Big]$$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}[y^{\beta}\int_0^{\infty}\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\{(\beta+1)-\frac{y(\alpha+\beta+1)}{(x+y)}\}\partial_{\ell,m}(x)dx]$$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}[y^{\beta}\int_0^{\infty}\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\{(\beta+1)-\frac{y(\alpha+\beta+1)}{(x+y)}\}\partial_{l,m}(x)dx]$$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta}\int_0^{\infty}\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+2}}\{x(\beta+1)-y\alpha\}\partial_{\ell,m}(x)dx.$$

Hence

$$y\frac{d}{dy}U_{\alpha,\beta,\ell,m}(y)=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^{\infty}\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+2}}\{x(\beta+1)-y\alpha\}\partial_{\ell,m}(x)dx. \quad (1.13)$$

Now, from the middle term of (1.6), we have

$$-(\ell+m)[U_{\alpha,\beta,\ell,m}(y)-U_{\alpha,\beta,\ell-1,m}(y)]=-(\ell+m)[\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^{\infty}\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\partial_{\ell,m}(x)dx$$

$$-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^{\infty}\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\{\ \partial_{\ell-1,m}(x)dx\ ] \quad (1.14)$$

$$=-(\ell+m)\left[\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^{\infty}\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\left\{\partial_{\ell,m}(x)-\partial_{\ell-1,m}(x)\right\}dx\right]$$

$$=-(\ell+m)[\ \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^{\infty}\frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}}\{-\frac{x}{\ell+m}\frac{d}{dx}\partial_{\ell,m}(x)\}dx\ ]$$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}y^{\beta+1}\int_0^{\infty}\frac{x^{\alpha}}{(x+y)^{\alpha+\beta+1}}d\partial_{l,m}(x)dx$$

Intergrating by parts, we see that the above experession becomes

$$= \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1}\left[\{\frac{x^{\alpha}}{(x+y)^{\alpha+\beta+1}} \partial_{\ell,m}(x)\}_0^{\infty} - \right.$$

$$\left. \int_0^{\infty} \frac{(x+y)^{\alpha+\beta+1}\alpha x^{\alpha-1} - x^{\alpha}(\alpha+\beta+1)(x+y)^{\alpha+\beta}}{(x+y)^{2(\alpha+\beta+1)}} \partial_{\ell,m}(x)dx \right]$$

$$= -\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1}\int_0^{\infty} \frac{x^{\alpha-1}\{\alpha(x+y) - x(\alpha+\beta+1)\}}{(x+y)^{\alpha+\beta+2}} \partial_{\ell,m}(x)dx$$

$$= -\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1}\int_0^{\infty} \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+2}} \{\alpha y - x(\beta+1)\}\partial_{\ell,m}(x)dx$$

$$= \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1}\int_0^{\infty} \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+2}} \{x(\beta+1) - \alpha y\}\partial_{\ell,m}(x)dx$$

the integrated term vanishing, since (1.10) holds. Since right hand side of (1.13) and (1.14) are the same, therefore the first equality of (1.6) follows.

By the first equation of (1.6)

$$y\frac{d}{dy}U_{\alpha,\beta,\ell,m}(y) = -(\ell+m)[U_{\alpha,\beta,\ell,m}(y) - U_{\alpha,\beta,\ell-1,m}(y)]$$

$$= -\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1}\int_0^{\infty} \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}} \{-x\frac{d}{dx}\partial_{\ell,m}(x)\}\,\mathrm{dx}$$

$$= \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} y^{\beta+1}\int_0^{\infty} \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}} \tau_{\ell,m}(x)dx.$$

$$= B_{\alpha,\beta,\ell,m}(y)$$

This proves the second equalityof (1.6)

2.

For any number p, used an index (exponent), and such that $p > 1$, we write $q = \frac{p}{p-1}$, so that p and q are conjugate indices in the sense of Holdr's inequality,

i.e. $\frac{1}{p}+\frac{1}{q}=1$. Morevoer, we define q to be if $p = 1$. We use $\{d,e,f...\}$ to denote a positive constant depending only on the parameters $d,e,f,...$ not necessarily on the two occurrences. Inequality of the form $L \leq c(d,e,f,....)R$ are to be interpreted as meaning "if the expression R is finite, then the expression L is finite and satisfies the inequality."

3. Known Results

The following results we due to Mishra and Singh [6].

Theorem A. If $\ell > n \geq 1, \alpha > 0,\ \beta > -1, m > -1, then$

$$s(x) \to (D,\alpha,\beta)(C,\ell-1,m),$$

whenever $s(x) \to s(D,\alpha,\beta)(C,n-1,m)$.

Theorem B. Let $\ell > n \geq 0, r \geq 0$ and m > - 1. Suppose that $s\ (x)$ is summale (C,r,m) to s and that $\int_1^\infty \frac{\partial_{n,m}(x)}{x^{\beta+2}}dx$ converges, then $s(x)$ is summable $(D,\alpha,\beta)(C,\ell,m)$ to S.

4. DEFINITION OF STRONG SUMMABILITY BASED ON $(D,\alpha,\beta)(C,\ell,m)$ SUMMABILITY METHODS

Suppose that $\alpha > 0, \beta > -1, \ell \geq, m \geq -1$ and $p \geq 1$, if

$$\int_0^x \left|U_{\alpha.\beta.\ell-1.m}(y) - s\right|^p dy = o(x) a\ n\ d\ x \to \infty \tag{4.1}$$

Then we say that the function s(x) summable $[(D,\alpha,\beta)(C,\ell,m]_p$ or strongly summable $(D,\alpha,\beta)(C,\ell,m)$ with index p, to the sum s, if

$$\int_0^x \left|U_{\alpha.\beta.\ell-1m}(y)\right|^p dy = O(x) \text{ as } x \to 0 \tag{4.2}$$

then we say that the function $s\ (x)$ is bounded $[(D,\alpha,\beta)(C,\ell,m)]_p$ or strongly bounded $(D,\alpha,\beta)(C,\ell,m_p$ with index p.

When $p = 1$, $[(D,\alpha,\beta)(C,\ell,m)]_p$ and $[D,\alpha,\beta)(C,\ell,m)]$ denote the same method. The summability $(D,\alpha,\beta)(C,\ell-1,m)$ may be regarded as the case $p=\infty$ of summability $[(D,\alpha,\beta)(C,\ell,m)]_p$ as is evident by the following equality

$$\lim_{p\to\infty}\{x^{-1}\int_0^x \left|U_{\alpha,\beta,\ell,m}(y)-s\right|^p dy\}^{1/p} = \sup \left|U_{\alpha,\beta,\ell-1,m}(y)-s\right|^p.$$

We many also regard boundendness $(D,\alpha,\beta)(C,\ell-1,m)$ as the case $p = \infty$ of boundedness $[(D,\alpha,\beta)(C,\ell-1,m)]_p$.

5. SOME ELEMENTARY RESULTS

Theorem 5.1: If the function $s\ (x)$ is summable $[(D,\alpha,\beta)(C,\ell,m)]_p$ to the sum s, where $1\le p\le\infty, \alpha>0, \beta>-l, m>-1 \& \ell\ge 1$, then it is summable $(D,\alpha,\beta)(C,\ell,m)$ to the same sum.

In order to prove this Theorem, we require the following Lemma.

Lemma 5.1: Let $\ell > n \ge 1, \alpha>0, \beta>-1, m>-1$, suppose that $I_{n-1,m}$ converges. Then for a given $x > 0$,

$$(I_{n-1,m} = \int_I^\infty \frac{\partial_{n-1,m}(x)}{x^{\beta+2}}dx)$$

$$U_{\alpha,\beta,\ell-1,m}(x) = \frac{\Gamma(\ell+m)}{\Gamma(\ell-n)\Gamma(n+m)} x^{-(\ell+m-1)}\int_0^x y^{n+m-1}(x-y)^{\ell-n-1}U_{\alpha,\beta,n-1,m}(y)dy \quad (5.1)$$

where (5.1) is to be taken as including the assertion that both sides exist. This Lemma has been proved by Mishra and Singh [6].

Proof of Theorem 5.1: We observed that, if the function $s(x)$ is summable$[D,\alpha,\beta)(C,\ell,m]_p$ to the sum s, for any p such that $1<p\le\infty$, then it is summable $[D,\alpha,\beta)(C,\ell,m]$ to s. We may therefore suppose that $p = 1$, i.e.,

$$\int_0^x \left|U_{\alpha,\beta,\ell-1,m}(y)-s\right|dy = o(x)$$

Further, we may suppose that s = 0. Then replacing ℓ by $\ell+1$ and n by n+1 in

Lemma 5.1 we have

$$U_{\alpha,\beta,\ell,m}(x)\frac{\Gamma(\ell+m+1)}{\Gamma(\ell-n)\Gamma(n+m+1)}x^{-(\ell+m)}\int_0^x y^{n+m}(x-y)^{\ell-n-1}U_{\alpha,\beta,n,m}(y)dy \quad (5.2)$$

Taking $n=\ell$-1 in (5.2), we have

$$U_{\alpha,\beta,\ell,m}(x)=(\ell+m)x^{-(\ell+m)}\int_0^x y^{m+\ell-1}U_{\alpha,\beta,\ell-1,m}(y)dy$$

Now

$$\left| U_{\alpha,\beta,\ell,m}(x)\right| \le (\ell+m)x^{-(\ell+m)}\int_0^x y^{m+\ell-1}\left| U_{\alpha,\beta,\ell-1,m}(y)\right| dy$$

Intergrating by parts, we have

$$\left|U_{\alpha,\beta,\ell,m}(x)\right| \le (\ell+m)x^{-(\ell+m)}\left[\{o(x)y^{m+\ell-1}\}_o^x+(m+\ell-1)\int_0^x o(x)y^{m+\ell-2}dy\right]$$

$$=o(1).$$

Hence $\quad U_{\alpha,\beta,\ell,m}(x)=o(1).$

This proves Theorem 5.1.

Theorem 5.2 : If the funntion $s(x)$ is summable $[(D,\alpha,\beta)(C,n,m)]_p$ to s, where $p>1,\ \ell>n+\dfrac{1}{p},n\ge 1,$ then it is summable $(D,\alpha,\beta,)(C,\ell-1,m)$ to the sum

Proof : We may evidently suppose that $s=0$. By Lemma 5.1, we have

$$U_{\alpha,\beta,\ell-1,m}(x)=\frac{\Gamma(\ell+m)}{\Gamma(\ell-n)\Gamma(n+m)}x^{-(\ell+m-1)}\int_0^x y^{n+m-1}(x-y)^{\ell-n-1}U_{\alpha,\beta,n-1,m}(y)dy$$

Now

$$\left|U_{\alpha,\beta,\ell-1,m}(x)\right|\le Cx^{-(\ell+m-1)}\int_0^x y^{n+m-1}(x-y)^{\ell-n-1}\left|U_{\alpha,\beta,n-1,m}(y)\right|dy$$

$$= Cx^{-(\ell+m-1)} \int_0^x \{y^{\frac{m+n-1}{p}} \; |U_{\alpha,\beta,n-1,m}(y)| \} \{y^{\frac{m+n-1}{q}} (x-y)^{\ell-n-1}\} dy$$

Applying Hoder's inequality with indices p and q, we have

$$|U_{\alpha,\beta,\ell-1,m}(x)| \le Cx^{-(\ell+m-1)} \{\int_0^x y^{n+m-1} \; |U_{\alpha,\beta,n-1,m}(y)|^p dy\}^{1/p} \{\int_0^x y^{n+m-1} (x-y)^{(\ell-n-1)q} dy\}^{1/q}$$

$$= C\{x^{-(\ell+m-1)} o(x)^{\frac{n+m-1}{p} + 1/p + \frac{n+m-1}{q} + (\ell-n-1) + 1/q}$$

$$= o(x^0) = o(1)$$

Provided that $\ell > n + 1/p, \; n \ge 1.$

Hence $U_{\alpha,\beta,\ell-1,m}(x) = o(1)$ as $x \to \infty$

This proves Theorem 5.2

Theorem 5.3 : Suppose that $\ell \ge 1. \; 1 \le p \le q \le \infty (p < \infty); \ell > n + 1/p - 1/q,$

where $>$ maybe replaced by $\le$ in the last inequality in the case, $1 < p, \; q < \infty;$

or $1 = p, \; q = \infty$.

Then: (I) $[(D,\alpha,\beta)(C,n,m.]_p \Rightarrow [(D,\alpha,\beta)(C,\ell,m)]_{q,}$

(II) for any s

$$\sup_x \{^{x^{-1}} \int_0^x \; |U_{\alpha,\beta,\ell-1,m}(y) - s|^q dy\}^{1/q} \le C(p,q,\ell,m)$$

$$\sup_x \{x^{-1} \int_0^x \; |U_{\alpha,\beta,n-1,m}(y) - s/^p dy\}^{1/p} \qquad (5.3)$$

The case $p = q = \infty$ is given by Theorem A.

For the proof of Theorem 5.3 we require few Lemmas.

Lemma 5.2 : Let $0 < \mu < 1,$

$$S(w) = \frac{1}{w} \int_0^w \phi(x) dx \qquad (\phi(x) \ge 0)$$

and $s = \sup_{0<w<\infty} S(w)$

Then $\int_0^w x^{-\mu}\phi(x)dx \le c(\mu)sw^{1-\mu}$

If we take, $s(w) = o(1)$, as $w \to \infty$, then

$$\int_0^w x^{-\mu}\phi(x)dx = o(w)^{1-\mu}.$$

This Lemma is given by Mishra [4].

Lemma 5.3: If $f(x) \ge 0$ and $f(x) \in L^p(0,y), p>1,$ and if

$$f_a(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt,$$

where $0<\alpha<\dfrac{1}{p}$ and $q = \dfrac{p}{1-p\alpha,}$ then $f_\alpha(x) \in L^q(o,y)$

and $\{\int_0^y \{f_\alpha(x)\}^q dx\}^{1/q} \le C\{\int_0^y \{f(x)\}^p dx\}^{1/p},$

where $C = C(p,\alpha) = C(p,q)$.

This Lemma has been proved by Hady, Little wood and Polya [2].

We pass now to the proof of Theorem 5.3, we may evidently suppose that s = 0. Since Theorem 5.3 (I) is a corollary of Theorem 5.3 (II), it is sufficient ot prove Theorem 5.3 (II) Let $\ell = n+\gamma$

Case (i) : $1 \le p \le q \le \infty$ and $\gamma > \dfrac{1}{p} - \dfrac{1}{q},$

Case (ii) : $1 < p < q < \infty$ and $\gamma = \dfrac{1}{p} - \dfrac{1}{q},$

Case (iiia): $1 < p < q = \infty$ and $\gamma > \dfrac{1}{p} - \dfrac{1}{q}$

Case (iiib) $1 = p < q = \infty$ and $\gamma \geq \frac{1}{p} - \frac{1}{q}$.

Proof of Case (i) Let S denote the superemum on the right of (5.3) suppose that $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$ and $0 < \delta < 1$, so that $r(\ell - n - 1) > -1$ $r \geq 1$.

By Lemma 5.1, We have

$$\left|U_{\alpha,\beta,\ell-1,m}(x)\right| \leq Cx^{-(\ell+m-1)} \int_0^x y^{n+m-1}(x-y)^{\ell-n-1} \left|U_{\alpha,\beta,n-1,m}(y)\right| dy$$

$$= Cx^{-(\ell+m-1)} \int_0^x \{(x-y)^{r(\ell-n-1)(1-\frac{1}{p})} y^{r(n+m-1+\delta)(1-\frac{1}{p})-\delta(1-\frac{1}{p})}\}\{y^{-\delta(\frac{1}{p}-\frac{1}{q})} \left| U_{\alpha,\beta,n-1,m}(y)\right|^{p(\frac{1}{p}-\frac{1}{q})}\}$$

$$\{x-y)^{r(\ell-n-1)\frac{1}{q}} y^{r(n+m-1+\delta)\frac{1}{q}-\frac{\delta}{q}} \left|U_{\alpha,\beta,n-1,m}(y)\right|^{p/q} dy\}$$

Applying Holder's inequality with indices $\frac{p}{p-1}, \frac{pq}{q-p}$ and q. we have

$$\left|U_{\alpha,\beta,\ell-1,m}(x)\right| \leq Cx^{-(\ell+m-1)} \{\int_0^x (x-y)^{r(\ell-n-1)} y^{r(n+m-1+\delta)-\delta} dy\}^{1-\frac{1}{p}} \{\int_0^x y^{-\delta} \left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{(\frac{1}{p}-\frac{1}{q})}$$

$$\{\int_0^x (x-y)^{r(\ell-n-1)} y^{r(n+m-1+\delta)-\delta} \left| U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{1/q}$$

$$= Cx^{-(\ell+m-1)} x^{\{r(\ell-n-1)+r(n+m-1+\delta)-\delta+1\}(1-\frac{1}{p})} x^{(-\delta+1)(\frac{1}{p}-\frac{1}{q})} S^{p(\frac{1}{p}-\frac{1}{q})} \{\int_0^x (x-y)^{r(\ell-n-1} y^{r(n+m-1+\delta)-\delta}$$

$$\left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{1/q} \text{ (by lemma 5.2)} \tag{5.4}$$

Hence

$$\left|U_{\alpha,\beta,\ell-1,m}(x)\right|^q \leq Cx^{-\{r(\ell+m-2+\delta)-\delta+1\}} S^{q-p} \{\int_0^x (x-y)^{r(\ell-n-1)} y^{r(n+m-1+\delta)-\delta} \left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}$$

$$\int_0^w \left| U_{\alpha,\beta,\ell-1,m}(x)\right|^q dx \leq CS^{q-p} \int_0^w \left[x^{-\{r(\ell+m-2+\delta)-\delta+1\}} \int_0^x (x-y)^{r(\ell-n-1)} y^{r(m-1+\delta)-\delta} \left|U_{\alpha,\beta,n-1,m}{}^{(y)}\right|^p dy\right] dx$$

Interchanging the order of integration, we have

$$\int_0^w \left| U_{\alpha.\beta.\ell-1,m}(x)\right|^q dx \le CS^{q-p} w^\delta \int_0^w (y^{r(n+m-1+\delta)-\delta} \left| U_{\alpha.\beta.n-1.m}(y)\right|^p \Big| dy \int_y^w (x-y)^{r(\ell-n-1)} (x)^{-r(\ell+m-2+\delta)-1} dx$$

$$\le CS^{q-p} w^\delta \int_0^w (y^{r(n+m-1+\delta)-\delta} \left|U_{\alpha.\beta.n-1.m}(y)\right|^p dy . y^{-r(n+m-1+\delta)}$$

$$\le CS^{q-p} w^\delta \int_0^w y^{-\delta} \Big|U_{\alpha.\beta.n-1.m}(y)/dy,$$

$$\le CS^{q-p} w^\delta S^p w^{1-\delta}, \text{ by Lemma. 5.2}$$

$$\le CS^q w$$

Since

$$\int_x^w (y-u)^n y^{-n-\ell-1} dy \le \int_u^\infty (y-u)^n y^{-n-\ell-1} dy$$

$$= \frac{u^{-\ell}\Gamma(n+1)\Gamma(\ell)}{\Gamma(\ell+n+1)}, \qquad (n>-1, u>0, \ell>0).$$

Since n+m-1>-1, we may choose δ such that r (n+m-1+δ)>0.

Thus by using Lemma 5.2, we have.

$$\sup_{w\ge 0}\{w^{-1}\int_0^w \left|U_{\alpha.\beta.\ell-1,m}(x)\right|^q dx\}^{1/q} \le C(p,q,n,m) \sup_{w\ge 0}\{w^{-1}\int_0^w \left|U_{\alpha.\beta.n-1.m}(y)\right|^p dy\}^{1/p}$$

When $p = 1$, The first factor of (5.4) is to be omitted. Similarly when $q = p$, the second brackted factor in (5.4) is to be omitted.

This proves Case (i)

Proof of Case (ii)

$$1 < p < q < \infty, \qquad 0 < \gamma = \frac{1}{p} - \frac{1}{q} < \frac{1}{p}$$

By Lemma 5.1, we have

$$\left|U_{\alpha,\beta,\ell-1,m}(x)\right| \le Cx^{-(n+\gamma+m-1)}\int_0^x y^{n+m-1}(x-y)^{\gamma-1}\left|U_{\alpha,\beta,n-1,m}(y)/dy\right.$$

$$= Cx^{-n-\gamma-m+1}\int_0^{x/2} y^{n+m-1}(x-y)^{\gamma-1}\left|U_{\alpha,\beta,n-1,m}(y)\right| dy + Cx^{-n-\gamma-m+1}\int_{x/2}^x (x-y)^{\gamma-1}y^{n+m-1}\left|U_{\alpha,\beta,n-1,m}(y)\right| dy$$

$= C(I_1 + I_2)$, say.

Hence, by Minkowski's inequality, it is sufficient to prove that

$$\sup_x \{x^{-1}\int_0^x \left|I_1\right|^q dy\}^{1/q} \le C\sup_x \{x^{-1}\int_0^x \left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{1/p} \quad (5.5)$$

and

$$\sup_x \{x^{-1}\int_0^x \left|I_2\right|^q dy\}^{1/q} \le C\sup_x \{x^{-1}\int_0^x \left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{1/p} \quad (5.6)$$

Proof of (5.5) :

Since $0<\gamma<1, (x-y)^{\gamma-1}$ Increases in (0,x/2), therefore we have

$$I_1 \le Cx^{-n-\gamma-m+1+\gamma-1}\int_0^{x/2} y^{n+m-1}\left|U_{\alpha,\beta,n-1,m}(y)\right|dy.$$

Proceeding as in the case (i), we see that (5.5) is established.

Proof of (5.6) :

$$I_2 \le C\int_{x/2}^x (x-y)^{\gamma-1}y^{-\gamma}\left|U_{\alpha,\beta,n-1,m}(y)\right|dy.$$

Applying Lemma 5.3, we have

$$\{\int_0^x \left|I_2\right|^q dy\}^{1/q} \le C\{\int_0^x y^{-(1-\frac{p}{q})}\left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{1/p}$$

Since $1-p/q<1$, therefore we can apply Lemma 5.2. Thus.

$$\sup_x \{x^{-1}\int_0^x \left|U_{\alpha,\beta,\ell-1,m}(y)\right|^q dy\}^{1/q} \le C(p,q,n)\sup_x \{x^{-1}\int_0^x \left|U_{\alpha,\beta,n-1,m}(y)\right|^p dy\}^{1/p}.$$

This proves the Case (ii)

Proof of Case (iiia) is given by Theorem5.2 Case (iiib) is given by Theorem 5.1. Thus, we have completely proved Theorem 5.3.

Theorem 5.4: The necessary and sufficient conditions for the function $s\ (x)$ to be summable $[(D,\alpha,\beta])(C,l,m)]_p$ to s are that it be summable $[(D,\alpha,\beta])(C,l,m)]_p$ to s and

$$\int_0^x \left| y\frac{d}{dy}U_{\alpha,\beta,\ell,m}(y)\right|^p dy = o(x) \text{ as } x \to \infty. \qquad (\ell > 1, m > -1).$$

Proof. This result is analogous to Hyslop's Theorem 3 [1] on strong Cesaro summability. Since summability $[(D,\alpha,\beta])(C,l,m)]_p$ implies summability $[(D,\alpha,\beta])(C,l,m)]$ it follows from (1.7) and Minkowski's inequality, that since $U_{\alpha,\beta,\ell,m}(y) - s = o(1)$ as $y \to \infty$.

$$\{\int_0^x \left|U_{\alpha,\beta,\ell-1}(y) - s\right|^p dy\}^{1/p} - \{\int_0^x \left|U_{\alpha,\beta,\ell,m}(y) - s\right|^p dy\}^{1/p} \le \frac{1}{|\ell+m|}\{\int_0^x \left| y\frac{d}{dy}U_{\alpha,\beta,\ell,m}(y)\right|^p dy\}^{1/p} \quad (5.7)$$

$$\le \{\int_0^x \left|U_{\alpha,\beta,\ell-1,m}(y) - s/^p dy\right\}^{1/p} + \{\int_0^x \left|U_{\alpha,\beta,\ell,m}(y) - s\right|^p dy\}^{1/p}$$

$$= o(x)^{1/p} + o(x)^{1/p}$$

$$= o(x)^{1/p}$$

This implies that $\int_0^x \left| y\frac{d}{dy}U_{\alpha,\beta,\ell,m}(y)\right|^p dy = o(x)$ as $x \to \infty$

Conversely suppose that function $s\ (x)$ is summable $[(D,\alpha,\beta])(C,l,m)]$ to s and

$$\int_0^x \left| y\frac{d}{dy}U_{\alpha,\beta,l,m(y)}\right|^p dy = d\ (x) \text{ as } x \to \infty$$

Again using (5.7), we have

$$\int_0^x \left|U_{\alpha,\beta,l-1,m}(y)-s\right|^p dy = o(x) \quad \text{as } x\to\infty$$

This completes the proof of Theorem 5.4.

6. RELATIVE STRENGTH BETWEEN $[D,\alpha,\beta)(C,\ell,m]_p$ AND $[C,n,m]_p$

Theorem 6.1: Suppose that $1\le p\le\infty$ and $n\ge 0$. If the function $s\ (x)$ is summable $[C,n,m]_p$ to the sum s, it summable $[D,\alpha,\beta)(C,\ell,m]_t$ to the sum s for every $t\ge 1$, $\ell\ge 1$. Wherever $(D,\alpha,\beta)\ (C,\ell-1,m)$ method is appliable.

The proof of this theorem follows immediately from B and the chain of implications

$[C,n,m]_p \Rightarrow (C,n.m)\Rightarrow (D,\alpha,\beta)(C,\ell-1,m)\Rightarrow [(D,\alpha,\beta)(C,\ell,m]_t$. for every $t=\ge 1$,

Theorem 6.2 Suppose $1\le p\le\infty$ and $\ell\ge 1, m>-1, \ell>\delta>0$. If the function s (x) is bounded $[(D,\alpha,\beta,)(C,\ell-\delta,m]_p$ and summable $[D,\alpha,\beta,)(C,\ell-\delta+1,m]_p$ to the sum s, it is summable $[D,\alpha,\beta,)(C,\ell,m]_p$ to s

Proof. We may suppose that $s=0,\ 0<\delta<1/p$. By Lemma 5.1 replacing n by $\ell-\delta$ we have

$$U_{\alpha,\beta,\ell-1,m}(x) = Cx^{-(\ell+m-1)}\int_0^x (x-y)^{\ell-\ell+\delta-1} y^{\ell-\delta+m-1} U_{\alpha,\beta,\ell-\delta-1,m}(y)dy$$

$$= Cx^{-(\ell+m-1)}\int_0^{x(1-\rho)} (x-y)^{\delta-1} y^{\ell-\delta+m-1} U_{\alpha,\beta,\ell-\delta-1,m}(y)dy + Cx^{-(\ell+m-1)}\int_{x(1-\delta)}^x (x-y)^{\delta-1} y^{\ell-\delta+m-1} U_{\alpha,\beta,\ell-\delta-1,m}(y)dy.$$

$= I_1(x)+I_2(x)$, say,

ρ being a fixed number such that $0<\rho<1/2$. Let now

$$S = \sup_x \{x^{-1}\int_0^x \left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right|^p dy\}^{1/p}. \tag{6.1}$$

We have $\left|I_2(x)\right|^p \le C\{x^{-(\ell+m-1)}\int_{x(1-\rho)}^x (x-y)^{\delta-1} y^{\ell-\delta+m-1} \left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right| dy\}^p$

$$\leq C\{\int_{x(1-\rho)}^{x} (x-y)^{\delta-1} y^{-\delta} \left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right| dy\}^{p}$$

or

$$\left|I_2(x)\right|^{p} \leq C\{\int_{x(1-\rho)}^{x} (x-y)^{\frac{\delta-1}{p}} y^{-\delta} (x-y)^{\frac{\delta-1}{q}} \left|U_{\alpha,\beta,\ell-1,m}(y)\right| dy\}^{p}$$

Applying Holder's inequality with indices p and q, we have

$$\left|I_2(x)\right|^{p} \leq C\{\int_{x(1-\rho)}^{x} (x-y)^{\delta-1} y^{-\delta p} \left|U_{\alpha,\beta,\ell-1,m}(y)\right|^{p} dy\}\{\int_{x(1-\rho)}^{x} (x-y)^{\delta-1} dy\}^{p/q}$$

$$= C(\ell,m,\delta,p) x^{\delta(p-1)} \rho^{\delta(p-1)} - \int_0^t \{x^{\delta(p-1)} \int_{x(1-\rho)}^{x} (x-y)^{\delta-1} y^{\delta p} \left| U_{\alpha,\beta,\ell-1,m}(y)\right|^{p} dy.$$

Hence

$$\int_0^t \left|I_2(x)\right|^{p} dx \leq C\rho^{\delta(p-1)} - \int_0^t \{x^{\delta(p-1)} \int_{x(1-\rho)}^{x} (x-y)^{\delta-1} y^{-\delta p} \left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right|^{p} dy\} dx$$

$$\leq Ct^{\delta(p-1)} \rho^{\delta(p-1)} \int_0^t dx \int_{x(1-\rho)}^{x} (x-y)^{\delta-1} y^{-\delta p} \left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right|^{p} dy.$$

Interchanging the order of integration, we have

$$\int_0^t \left|I_2(x)\right|^{p} dx \leq Ct^{\delta(p-1)} \rho^{\delta(p-1)} \int_0^t y^{-\delta p} \left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right|^{p} dy \int_y^{y/1-\rho} (x-y)^{\delta-1} dx$$

$$= Ct^{\delta p} \rho^{\delta p} (1-\rho)^{\delta} \int_0^t y^{-\delta p} \left|U_{\alpha,\beta,\ell-\delta-1,m}(y)\right|^{p} dy$$

Applying Lemma 5.2 we have

$$\int_0^t \left|I_2(x)\right|^{p} dx \leq CS^{p} \rho^{\delta p} (1-\rho)^{-\delta} t,$$

(since $\delta p < 1$), the constant being independent of ρ. Replacing ℓ by $n+1$ in Lemma 5.1, we have

$$U_{\alpha,\beta,n,m}(x) = Cx^{-(n+m)} \int_0^x y^{n+m-1} (x-y)^{n+1-n-1} U_{\alpha,\beta,n-1,m}(y)dy$$

or

$$U_{\alpha,\beta,n,m}(x) = Cx^{-(n+m)} \int_0^x y^{n+m-1} U_{\alpha,\beta,n-1,m}(y)dy$$

we have

$$I_1(x) = Cx^{-(\ell+m-1)} \int_0^{x(1-\rho)} (x-y)^{\delta-1} y^{\ell-\delta+m-1} U_{\alpha,\beta,\ell-\delta-1,m}(y)dy$$

Integrating by parts

$$I_1(x) = Cx^{-(\ell+m-1)}[(x-y)^{\delta-1} y^{\ell-\delta+m} U_{\alpha,\beta,\ell-\delta,m}(y)]_0^{x(1-\rho)} + Cx^{-(\ell+m-1)} \int_0^{x(1-\rho)} (\delta-1)(x-y)^{\delta-2} y^{\ell-\delta+m} U_{\alpha,\beta,\ell-\delta,m}(y)dy$$

$$= Cx^{-(\ell+m-1)}[(\rho x)^{\delta-1} x^{\ell-\delta+m}(1-\rho)^{\ell-\delta+m} U_{\alpha,\beta,\ell-\delta,m}\{x(1-\rho)\}] + Cx^{-(\ell+m-1)}(\delta-1)\int_0^{x(1-\rho)} (x-y)^{\delta-2} y^{\ell-\delta+m} U_{\alpha,\beta,\ell-\delta,m}(y)dy$$

$$= C(1-\rho)^{\ell-\delta+m} \rho^{\delta-1} U_{\alpha,\beta,\ell-\delta,m}\{x(1-\rho)\} + Cx^{-(\ell+m-1)} \int_0^{x(1-\rho)} (x-y)^{\delta-2} y^{\ell-\delta+m} U_{\alpha,\beta,\ell-\delta,m}(y)dy$$

$$= I_{1,1}(x) + I_{1,2}(x), (say).$$

$$I_{1,1}(x) = C\rho^{\delta-1}(1-\rho)^{\ell-\delta+m} U_{\alpha,\beta,\ell-\delta,m}\{x(1-\rho)\}$$

$$| I_{1,1}(x)|^p \le C\rho^{\delta p-p}(1-\rho)^{p(\ell-\delta+m)} | U_{\alpha,\beta,\ell-\delta,m}\{x(1-\rho)\}|^p$$

Integration both sidess, we get

$$\int_0^t \left|I_{1,1}(x)\right|^p dx \le C(\ell,m,\delta,\rho,p)\int_0^t \left|U_{\alpha,\beta,\ell-\delta,m}\{x(I-\rho)\}\right|^p dx \tag{6.2}$$

$$= o\,(t).$$

$$\left|I_{1,2}(x)\right|^p = C\left|x^{-(\ell+m-1)} \int_0^{x(1-\rho)} (x-y)^{\delta-2} y^{\ell-\delta+m} U_{\alpha,\beta,\ell-\delta,m}(y)dy\right|^p$$

if $p > 1$ we have, by Holder's inequality with indices p and q,

$$\left|I_{1,2}(x)\right|^p \le Cx^{-(\ell+m-1)p}\{\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy\}\{\int_0^{x(1-\rho)} \{(x-y)^{\delta-2}\}^q dy\}^{p/q}$$

$$\left|I_{1,2}(x)\right|^p \le Cx^{-(\ell+m-1)p}\{\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy\Big[\{\frac{-(x-y)^{(\delta-2)q+1}}{(\delta-2)q+1}\Big]_0^{x(1-\rho)}\}^{p/q}$$

$$\le Cx^{-(\ell+m-1)p}\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy\left\{\frac{-[(\rho x)^{(\delta-2)q+1}-x^{(\delta-2)q+1}]^{p/q}}{[(\delta-2)q+1]^{p/q}}\right\}$$

$$\le Cx^{-(\ell+m-1)p}\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy\left\{\frac{x^{\{(\delta-2)q+1\}p/q}[\rho^{(\delta-2)q+1}-1]^{p/q}}{[(\delta-2)q+1]^{p/q}}\right\}$$

$$\le Cx^{-(\ell+m-1)p}\int_0^{x(1-p)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy\,x^{\delta p-p-1} \quad (\text{since } 1/p+1/q=1)$$

$$\le Cx^{-(\ell+m-1)p+\delta p-p-1}\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy$$

$$\le Cx^{-\ell p-mp+\delta p-1}\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p y^{(\ell-\delta+m)p}dy$$

$$\le Cx^{-1}\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right|^p dy$$

$$== O(x^{-1})o(x) = o(1).$$

If $p = 1$, we have

$$\left|I_{1,2}(x)\right| \le O(x^{-\ell-m+\delta-1}\int_0^{x(1-\rho)} \left|U_{\alpha,\beta,\ell-\delta,m}(y)\right| y^{\ell-s+m}\,dy.$$

$$\le O(x^{-\ell-m+\delta-1})o(x^{\ell-\delta+m+1})$$

$$= o(1),$$

and so in either case $\int_0^t \left|I_{1,2}(x)\right|^p dx \le C\int_0^t o(1)dx. = o(t)$ (6.3)

It follows by Minkowski's inequality that

$$\limsup \left\{t^{-1}\int_0^t \left|U_{\alpha,\beta,\ell-1,m}(x)^p\right|dx\right\}^{1/p}$$

$$\leq SC(\ell,m,\delta,p)\rho^{\delta}(1-\rho)^{-\delta/p}+o(1)+o(1)$$

$$\leq SC(\ell,m,\delta,p)\rho^{\delta}(1-\rho)^{-\delta/p} \tag{6.4}$$

Making $\rho\to 0$, it follows now that limit sup on the left of (6.4) is 0, and this complets the proof of Theorem. 6.2.

Theorem 6.3: Suppose that $1<p\leq\infty,\alpha>0,\beta\geq -1,m>-1,\ell>1$ and (1.5) holds. If the function $s\,(x)$ is summable $[D,\alpha,\beta)(C,\ell,m]_p$ to the sum s, then the function x a (x) is summable $[(D,\alpha,\beta)(C,\ell+1,m)]_p$ to the sum 0.

Theorem 6.4 :Suppose that $1\leq p\leq\infty,\alpha>0,\beta\geq -1,$ and (1.5) holds. If the function $s(x)$ is summable $[(D,\alpha,\beta)(C,\ell+1,m]_p$ to s and $xa(x)$ is summable $[(D,\alpha,\beta)(C,\ell+1,m)]_p$ to the sum 0, then the function $s(x)$ is summable $[(D,\alpha,\beta)(C,\ell,m)]_p$

Theorem 6.5: Suppose that $1\leq p\leq\infty,\alpha>0,\beta\geq -1,\delta>0$ and (1.5) holds If the function $s\,(x)$ is summable $[(D,\alpha,\beta)(C,\ell-\delta,m)]_p$ to s and the function $xa(x)$ is bounded $[(D,\alpha,\beta)(C,\ell-\delta,m)]_p$ then the function $s(x)$ is summable $[(D,\alpha,\beta)(C,1-\delta,m)]_p$ to the sum s.

Proof of Theorem 6.3 follows immediately Theorem 5.4 while Theorem 5.4 follows directly from (1.6) and Minkowski's inequality.

To prove Theorem 6.5, we first note that if the function $s(x)$ is summable $[(D,\alpha,\beta)(C,\ell-\delta,m)]_p$ then the function $xa(x)$ is summable $[(D,\alpha,1\beta)(C,\ell-\delta+1,m)]_p$. If also the function $xa(x)$ is bounded $[(D,\alpha,\beta)(C,\ell-\delta,m)]_p$, then it is summable $[(D,\alpha,\beta)(C,\ell,m)]_p$ to 0, and since summability$[(D,\alpha,\beta)(C,\ell-\delta,m)]_p$ implies summability $[(D,\alpha,\beta)(C,\ell,m)]_p$, Theorem 6.4 completes the proof of Theorem 6.5.

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EFFECT OF POLARIZABILITY ON NUCLEATION PHENOMENON DURING ICE GLACIATION DUE TO EXTERNAL ELECTRIC FIELD

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ABSTRACT

The polarizability of water vapour molecules in presence of external electric field plays an important role in the nucleation rate of water vapour condensation and ice glaciation. The polarizability decreases with increase in the temperature and hence there is decrease in Gibb's free energy and hence nucleation rate is decreased .Thus polarizability controls the ice glaciation in clouds.

Keywords and phrases: Polarization. Nucleation rate ,Gibb's free energy, Critical nuclei.

INTRODUCTION

The polarizability plays a dominant role in inducing the electric dipole on water vapour molecules. The contribution to polarization of water vapour molecules coming from the electric field generated by the droplet dipole, which increases the interaction between droplet and water vapour molecules. This would enhance the effect described by Murino [18], but the interaction due to this additional factor would not be steady and hence would produce only transient effect.

Ice nucleation is a fundamental cloud process. The process involved in ice formation is understood only in the case of homogeneous freezing of supercooled droplets. In contrast, ice formation caused by heterogeneous freezing nuclei at low temperatures are poorly understood. Kärcher and Lohmann [12] considered theoretically the nucleation and initial growth of ice crystals in cerrus clouds at

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low temperatures (< 235K) prevailing in the upper troposphere and in the tropopause region.

Ice crystals are produced by electric fields in cloud chambers [8] caused by fragmentation of dendritic crystals. The nucleation of ice in supercooled water by electric fields has been demonstrated. Nucleation in the thin liquid filaments, which form during disruption of drops or during movement of drops on surfaces was suggested by Loeb [13] and by Abbas and Latham [1] as a possible mechanism.

Nucleation as a result of distortion of drops by electric forces was reported [1,20,21,28]. When charged, aerosols and other surfaces coming in contact with drops were found to cause nucleation more readily than if no electric charge were present [9,21,23]. The ice formation in the atmosphere is a heterogeneous process initiated by ice forming nuclei [15].

Dufour [7] suggested that freezing could be initiated in supercooled water by means of an electric field. Unfortunately he gave only an insufficient description of his experiment so that no firm conclusions can be drawn. Dufour [7], Rau [22], Salt [25], Gabarashvili and Gliki [9] proposed that electro-freezing of supercooled water is due to dielectric polarization of water in high electric fields.

Collins [4] inferred that the relaxation time is independent of the free energy of fonnation of the nucleus, but it varies as the square of the radius of the critical nucleus. Bartlett et al. [3], as well as Maybank and Barthakur [16] reported the tendency of generation of a few ice splinters during ice crystal growth in strong electric fields. Doolittle and Vali [6] pointed out the effect of an electric field on samples of water at different time temperature cycles.

The formulation of relaxation time required for the attainment of the steady state concentrations of the embryos of the critical size has been discussed [4,11,19,29]. The scavenging of aerosol particles by hydro-meteors under external electric field has been discussed by Wang [30]. Sharma et al. [26] reported that the small values of electric field is equivalent to very high supersaturation ratio to get a nucleus of given size under similar conditions of temperature.

Singh et al. [27] have shown that in the resultant electric effect on a droplet

due to an external electric field and the field induced due to the central dipole, the role of nucleation in water vapour condensation and ice glaciation is about hundred times more near breakdown for dry air as compared to that in absence of electric field. Thunderstorm electric field promote droplet growth.

Electric field near triggered lightning channels have been measured with Pockels sensors. Mikki et al. [17] at the International Centre for Lightning Research and Testing at Camp Blanding, Florida. The vertical electric field pulse peaks are in the range from 176 kV/m to 1.5 MV/m (the median is 577 kV/m) and horizontal electric field pulse peaks are in the range from 495 kV/m to 1.2 MV/m (the median is 821 kV/m). The vertical electric field measured very close to the lightning channel tends to increase with an'increase in the previous no current interval.

The atmospheric electric field and the electrical conductivity varies from place to place depending upon certain environmental factors. Deshpande and Kamra [5] gave the surface measurements of the atmospheric electric potential gradient and conductivity at Indian station, Maitri (70^0 45' 52" S, $11^0$44' 3" E, 117 m above mean sea level), Antarctica, from January 10 to February 24, 1997. The diurnal variation of potential gradient averaged for 20 fair weather days is single periodic with maximum at 1300 UT and another secondary but distinct maximum at 1900 UT and a minimum at 0100 UT. The mean value or the potential gradient is 83 V/m and the maximum and minimum values are 1.55 and 0.73 times the mean value, respectively. The total conductivity is 2.1×10^{-14} S/m and does not show any significant diurnal variation. The diurnal variation of the potential gradient observed at Maitri is similar to the one observed a few days later in the southern Indian Ocean and to the one reported by Kamra et al. [10] with a few hours difference in phase in the northern Indian Ocean.

Rust and Trapp [24] presented the first known vertical profiles of electric field in six winter nimbo stratus clouds in the U.S.A. The maximum magnitude of the vertical component of electric field in the profiles ranged from 1 to 12 kV/m, the maximum horizontal component ranged from 0.2 to 28 kV/m.

Marshall and Stolzenburg [14] used a one-dimensional model to calculate

the electrostatic energy of several horizontally the extensive, electrified clouds. Model estimates of the model calculation are initiated with known charge distributions. Small reductions in the charge density of a region often can provide a relatively large amount of energy to derive a flash. Model calculations suggest that positive cloud-to-ground flashes that produce Q-bursts have energies of about 1×10^{10} J. Total electrostatic energy stored in two stratiform clouds of mesoscale convective systems were 5×10^{11} J and 2×10^{12} J. These energies are sufficient to support hundreds or thousands of typical lightning flashes, but only 10-100 of the energetic positive cloud-to-ground flashes with associated Q-bursts.

To authors knowledge the role of effective polarizability has not been considered affecting nucleation rate of ice particles. In present study, we have evaluated the effect of polarizability on nucleation effective polarizability. We estimated nucleation rate of ice particles in external electric field using modified value of polarizability at different temperatures. Due to the effective polarizability α_{eff}, the Gibb's free energy, nucleation rate and radius of nucleus of ice are found to increase.

THEORETICAL CONSIDERATION

The effective or resultant polarizability may be written as

$$\alpha_{eff} = \alpha + \frac{p_0^2}{3kT} \tag{1}$$

Thus, effective polarizability varies inversely as absolute temperature of medium. In an external electric field, the moment induced on a droplet is

$$\vec{M} = \vec{E} r_i^3 \tag{2}$$

where, r_i is the radius of ice embryo (assumed spherical) and $\vec{E}$, the electric field.

The moment induced on water vapour molecule, present in the cloud is

$$\vec{M}' = \alpha \vec{E} \tag{3}$$

where, α is the polarizability. For α we assume the value 5×10^{-23} cm^3, observing that the contribution from the permanent electric dipole to polarizability is

$$\frac{p_0^2}{3kT} = 2.6 \times 10^{-23} \text{ cm}^3$$

where, p is the permanent dipole moment; k, the Boltzmann constant and T, the absolute temperature.

HOMOGENEOUS NUCLEATION

In the classical models of drop condensation and ice nucleation the Gibbs function has been taken into consideration. However, Abraham[2] has shown that the Helmholtz free energy is the proper thermodynamic potential and Gibbs function is only its approximation.

The Gibb's free energy for formation of a ice embryo is given by

$$\Delta G_i = -\frac{4}{3}\pi r_i^3 \Delta G_v + 4\pi r_i^2 \sigma_{i/v} \tag{4}$$

where $\Delta G_v = \dfrac{\rho_i RT \ln S_{v.i}}{M_w}$

R, the universal Gas constant; $S_{v.i}$, the supersaturation ratio of water vapour over ice surface; M_w, the molecular weight of water.

Critical radius of ice nucleus is given by

$$\frac{\partial \Delta G_i}{\partial r_i} = 0$$

The radius of critical ice nucleus is

$$r_i^* = 2\sigma_{i/v} M_w / \rho_i RT \ln S_{v.i} \tag{5}$$

The numher ice molecules in a critical nucleus is given by

$$n_i^* = 4\pi r_i^{*3}\rho_i N / 3M_w \tag{6}$$

where, ρ is density of ice; N, the Avogadro number.

The energy of formation of a critical nucleus is

$$\Delta G_i^* = 4\pi r_i^{*2}\sigma_{i/v} / 3 \tag{7}$$

The nucleation rate of ice nuclei is

$$J_i^* = \text{a constt.}\exp\left[-\Delta G_i^* / kT\right] \tag{8}$$

IN PRESENCE OF EXTERNAL ELECTRIC FIELD

Water is strongly polarizable having dipole moment 1.83×10^{-18} esu cm. In an external electric field, the water droplet (or embryo) rapidly increases in size. The expression for the rate of growth of radius of ice embryo has been derived as [27].

$$\frac{dr_i'}{dt} = \frac{\rho_v}{\rho_i}\left[9\alpha_{eff}\lambda E^2 / m_w r_i'\right]^{\frac{1}{2}} \tag{9}$$

where, λ is the mean free path; E, the external electric field; m_w, the mass of water vapour molecule; ρ_v, the density of water vapour and r_i' the radius of ice and embryo in the electric field. Integrating above equation within limits $r_i' = 0$ to $r_i'^*$ (critical radius of ice nucleus) and $t = 0$ to $t = \tau$ (relaxation time)

$$r_i'^* = \left[3\rho_v\tau\left(9\alpha_{eff}\lambda E^2 / m_w\right)^{1/2} / 2\rho_i\right]^{2/3} \tag{10}$$

Analogically, in the external field eqn. (3.32) and (3.33) for the energy of formation and the nucleation rate of ice nucleus, becomes

$$\Delta G_i^{'*} = \frac{4\pi}{3} r_i^{'*} \sigma_{i/v} \quad (11)$$

and

$$J_i^{'} = \text{a constant. } \exp\left(-\frac{\Delta G_i^{'*}}{kT}\right) \quad (12)$$

RESULTS

The effective polarizability α_{eff}, of water varies nearly inversely as the absolute temperature. The variation of α_{eff} with temperature has been shown in table 1.

Table 1. : Calculated values of α_{eff} of water molecules with the absolute temperature using $\alpha = 5x10^{-23}$ cm^3.

T(K)	$\alpha_{eff}(x10^{-23})$ cm^3
263	8.006
273	7.896
283	7.794
293	7.698
303	7.609

Using eqn. (9), (10) and (11) the critical radius of the ice nucleus r_i^*, Gibb's free energy of formation ΔG_i^* and natural logarithm of nucleation rate $ln\ J_i^*$ as the function of external electric field at relaxation time 2 x 10^{-6} see at temperatures 263, 273 and 283 K have been shown in Table 2.

Calculated value of critical radius of ice, r_i^*, Gibb's free energy of formation of critical ice nucleus, ΔG_i^*, and natural logarithm of nucleation rate of ice nuclei $ln\ J_i^*$ as the function of the relaxation time at constant electric field 10 esu at temperatures 263, 273 and 283 K are given in Table 3.

Table 2. : Calculated values of $r_i^*, \Delta G_i^*$ and In J_i^* as the function of external electric field at $\tau = 2\times10^{-6}$ s and temperatures 263, 173 and 283 K.

E (esu)	$r_i^*(\times10^{-9})$cm	$\Delta G_i^*(\times10^{-14})$erg	$\ln J_i^*$
		T = 263 K	
5	18.59	14.48	3.99
10	29.50	36.48	10.05
15	38.66	62.63	17.26
		T = 273 K	
5	18.51	14.35	3.81
10	29.38	36.16	9.60
15	38.49	62.09	16.48
		T = 283 K	
5	18.43	14.23	3.64
10	29.25	35.85	9.18
15	38.33	61.55	15.76

Table 3. : Calculated values of $r_i^*, \Delta G_i^*$ and In J_i^* as the function of relaxation time at E = 10 esu at temperatures 263, 173 and 283 K.

T $(\times10^{-9})$sec	$r_i^*(\times10^{-9})$cm	$\Delta G_i^*(\times10^{-14})$erg	$\ln J_i^*$
		T = 263 K	
10	86.27	30.92	82.94
20	136.94	78.58	216.52
30	179.44	134.93	371.78
		T = 273 K	
10	85.90	30.65	82.06
20	136.35	76.23	206.79
30	178.67	133.77	355.08
		T = 283 K	
10	85.52	30.10	78.48
20	135.76	73.91	197.75
30	177.89	132.61	339.56

DISCUSSION

The calculations in Table l, 2 and 3 have been made using the following constants:

ρ_v = 10^{-5} at 10^0 gm cm^{-3}

α = 5.0×10^{-23} cm^3

p = 1.81×10^{-18} esu

k = 1.38×10^{-16} erg K^{-1}

λ = 10^{-5} cm

m_w = 3.0×10^{-23} gm

M_w = 18

ρ_i = 0.917 gm cm^{-3}

$\rho_{i/v}$ = 100 dynes cm^{-1}

R = 8.317×10^7 erg K^{-1} mole^{-1}

N_A = 6.025×10^{23}

The effective polarizability of water varies approximately inversely as absolute temperature(Table 1). From Table 2 it is clear that radius decreases with increase in temperature. Also Gibb's free energy decreases with increase in temperature.

From Table 2, we see that at the given temperature, the value of $r_i^*, \Delta G_i^*$ and $\ln J_i^*$ are found to increase with increase in applied external electric field. But, at a fixed external electric field, the values of $r_i^*; \Delta G_i^*$ and $\ln J_i^*$ decrease with increase in temperature.

For example. at electric field 10 esu, the radii are 29.50×10^{-9}, 29.38×10^{-9} and 29.25×10^{-9} cm corresponding to the temperatures 263, 273 and 283 K, respectively. Similarly, at the same electric field, the critical Gibb's free energy for formation of ice nucleus are 36.48×10^{-14}, 36.16×10^{-14}, and 35.85×10^{-14} erg at temperatures 263, 273 and 283 K, respectively. Similar variations are evident for $ln\ J_i^*$.

Table 3 represents the calculated values of $r_i^*, \Delta G_i^*$ and $\ln J_i^*$ as the function of relaxation time at electric field 10 esu for temperature 263, 273 and 283 K. These values have been found to increase with increase in relaxation time for given temperature. But at a given relaxation time, these value decrease with increase in temperature.

From above discussion we may conclude that the nucleation parameters; critical radius. Gibb's free energy are highly decreased, while nucleation rate is increased with increase in temperature. Thus, the nuclei of smaller size requiring less Gibb's free energy are produced in large numbers i.e. nucleation rate is enhanced.

CONCLUSION

From above study, it may be concluded that the nucleation process is accelerated in presence of external electric field as compared with the homogeneous nucleation process. In clouds and thunderstorms, electric fields are generated due to lightning discharges.

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KĀTYĀYNA ŚULB-SŪTRA

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ABSTRACT

Mature Science offers a new tool of interpretation to the histrorians who is then able to discover new results in those original works that they have already studied. Here Kandika 3 of Kātyāyana Śulba Sūtra deals with some gemetrical figures and their transformations.

Key Words : KSS- Kātyāyana Śulba Sūtra, KSS-3/1 - first Verse of their Kandika of KSS.

INTRODUCTION

This paper aims at the conversion of area i.e. the conversion from rectangle into square and vice-versa ; circle in to square and vice-versa of various dimensions. It focuses how the four sided complicated structures have been simplified in the Śulba Sūtra. Here the endeavor has also been made to highlight the method of construction and proof of each sutra. This paper is discussed under the following topics.

- Extraction of a small square from large one.
- Conversion of a Rectangle into square.
- Transformation of a Lengthy rectengle in to a square.
- Transformation of a square in to rectangle.
- Some triplets for areas.
- Transformation of square in to circle and circle in to suquare.

EXTRACTION OF A SMALL SQURE FORM LARGE ONE

चतुरस्राच्चतुरस्रं निर्जिहीर्षन् यावन्निर्जिहीर्षे त्तावदुभयतोऽपच्छिद्य शङ्कू
निखाय पार्श्वमानी कृत्वा पार्श्वमानी सम्मितामक्ष्णयां तत्रोपसंहरति
समासेऽपच्छेद: सा करण्येष निह्रसि: ॥ 3/1 ॥

This means mark the length of the side of a small square on two opposite sides of a big one. Join the two points by a straight line Fix one point of straight line as center and other as moving point Extend the moving point to the base of big square. Now the large portion of base will be side of a required square which will be remaining area after subtracting the small square form the large one. Thus, small square, EFGH has been extracted form the given large square ABCD

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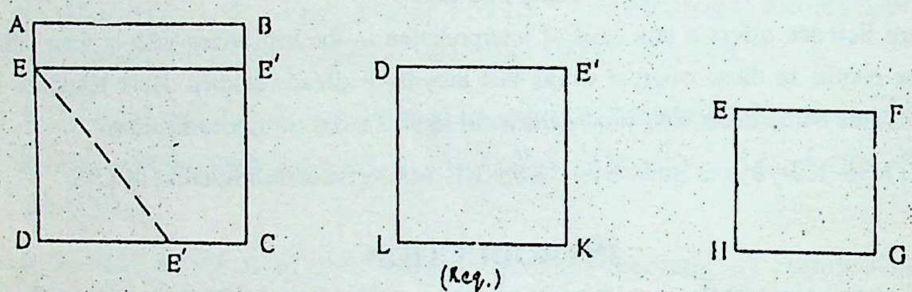
In right-angled triangle EDE'

$ED^2 + DE'^2 = EE'^2$

$ED^2 + EF^2 = AB^2$ $(EE' = AB, DE' = EF)$

$ED^2 = AB^2 - EF^2$



CONVERSION OF A RECTANGLE INTO A SQUARE

दीर्घचतुरस्रं समचतुरस्रं चिकीर्षन्मध्ये तिर्यगपच्छिद्यान्यतरद्विभज्ये

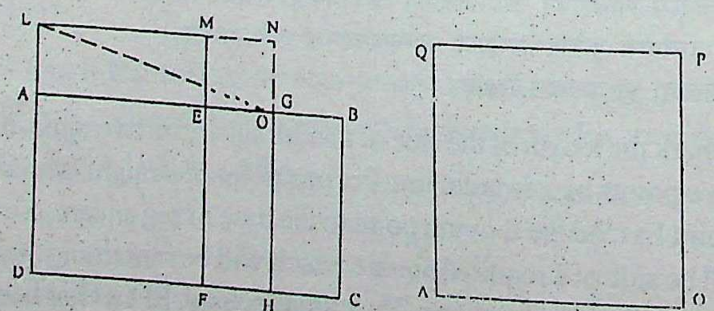
तरत्पुरं-स्ताद्दक्षिणतश्चोपदध्याच्छेषमागन्तुना पूरयेत्तस्योक्तो निर्ह्रास ॥ 3/2 ॥

To Transfer a rectangle ABCD into square draw in line EF bisecting the lines AB and DC. Now in Sq. EBCF draw a line GH bisecting sides EB and FC. Place rect. GBCH at AE and we have LMEA. Now Complete a big square LNHD by adding sq. MNGE.

Mathematically

$AO^2 = OL^2 - AL^2$ $(OL = LN)$

$AO^2 = LN^2 - AL^2$



Thus AO^2 represent the square AOPQ. Here square MNGE is being subtracted form the square LNHD to get the required square AOPQ.

TRANSFORMATION OF A LENGTHY RECTENGLE IN TO A SQUARE

अतिदीर्घञ्चे चिर्यङ् मान्याऽपच्छिद्यैक समासेन समस्य शेषं यथा योगमुपसहरे दिदत्येक समासः ॥ ३/३॥

A long rectangle ABCD is divided in to equal squares ABOL, LOPM, MPQN and one small rectangle DCQN.

In fig. 2 a square will represent area of three squares (which is made on the hypotenuse of right angled triangle.)

Cut an arc GT = FC in GR taking G as center. Join OT.

In fig. 3 a square made on OT will represent the req. area of given long rectangle.

i.e. $AB^2 + LO^2 + MP^2 + NF^2 + FQ^2 = OT^2$

Since $AB^2 + LO^2 + MP^2 = OG^2$

Again in Δ OGT

$OG^2 + GT^2 = OT^2$

$(GT = CF)$

$(CF^2 = FQ^2 + NF^2)$

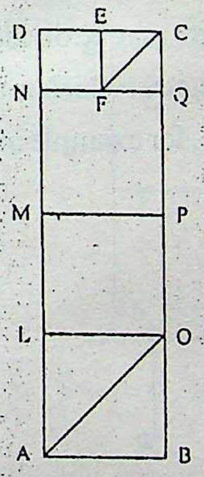


Fig. 1

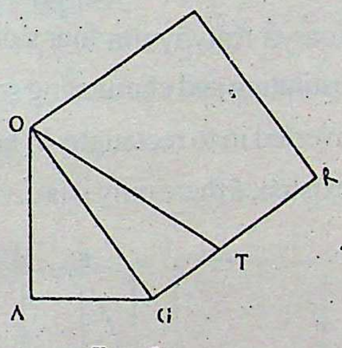


Fig. 2

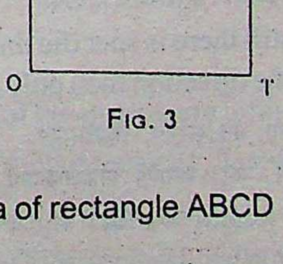


Fig. 3

Hence area of required square OTUV = area of rectangle ABCD

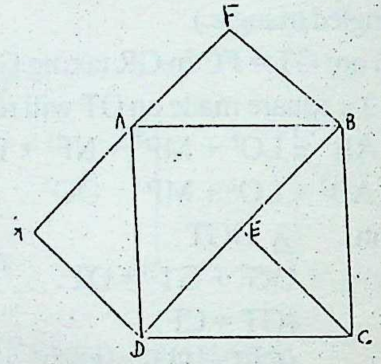
TRANSFORATION OF A SQUARE IN TO RECTANGLE

समचतुरस्रं दीर्घचतुरस्रं चिकीर्षन्मध्येऽक्ष्णयाऽपच्छिद्य विभज्येतरत्

पुरस्तादुत्तरतश्चोपदध्याद, विषमं चेद् यथा-योगमुपसंहरेदिति व्यस: ॥ 3/4 ॥

Divide a square by diagonal into two equal parts. Again divide one of the parts in to two so that we have two isosceles Δs

Place these two triangles on the opposite corners of the adjacent sides.



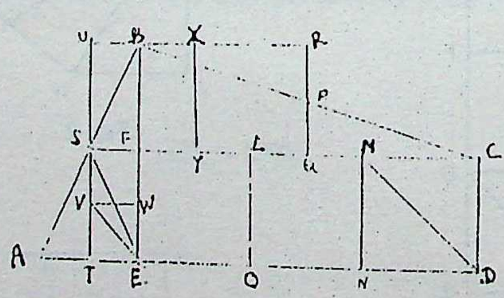
Δ DEC $\equiv$ Δ BEC

Place Δ BEC on Δ FAB.

Similarly put Δ EDC on Δ GDA.

GDBF is a required rectangle.

Some examples have been shown to describe this situation by Ācāryā Kātyāyana. According to Ācāryā Kātyāyana four sidea figure (quadrilateral) can be converted in to rectangle by substituting and eliminating some area. The figure may be divided in to small square then converted in to rectangle. Where the required figure (which is rectangle) do not come in to existence there only it is divided in to squares, for example in figure 1.



SOME TRIPLETS FOR AREA

प्रमाणं चतुरसमादेशादन्यत् ॥ 3/5 ॥

Unless or untill there is specific direction of unit, a square unit is to be accepted. If there is some specific direction then only act accordingly.

द्वि प्रमाणा चतुःकरणी, त्रिप्रामाणा

नवकरणी, चतुः प्रमाणा षोडशकरणी ॥ 3/6॥

As much the length so much squared area will be got, because same diglts will be multipiled with each other. If measure of unit is two then its area will be four unit of square,

If three then nine, if four then 16 and so on. Thus this is a little told about area.

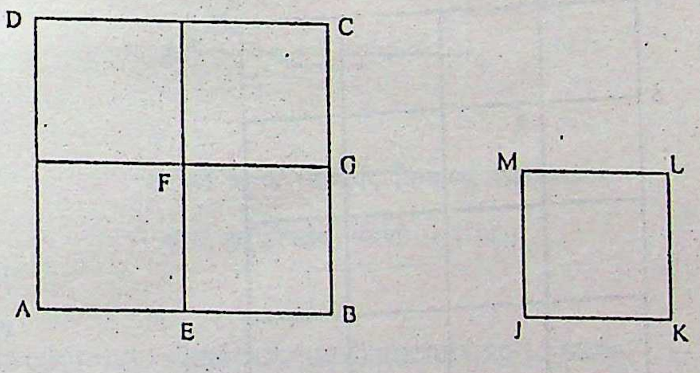
यावत् प्रमाणा रज्जुर्भुवति तावन्तस्तावन्तो

वर्गा भवन्ति तान्समस्येत् ॥ 3/7 ॥

As many units of measure in a chord so many squares of those units will be. Add all those squares then area will be got. If two units of measure taken, four squares will be formed for area (2×2 = 4). If five units are there then 25 squares wil be formed (5×5 = 25). Similarly other can be produced.

अर्धप्रमाणेन पादप्रमाणं, विधीयते ॥ 3/8 ॥

But when half measure is taken then what will square of that space be ? If half the measure is squared it is fourth part of complete spare $(1/2)^2 = 1/4$. If the measure is 24 units. Its half is 12 Square of 24 is 576 and that of 12 is 144. Thus 144 is one fourth of 576. This way for other units of measure can be applied.

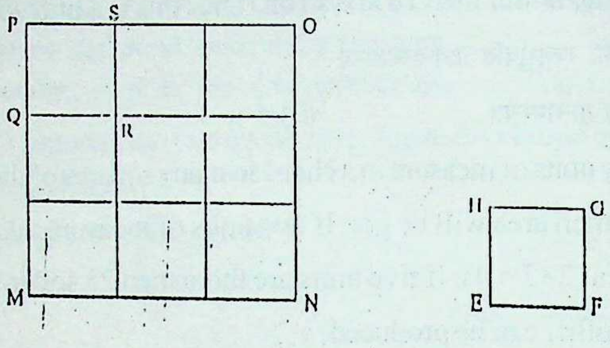


Example

Let us take ABCD whose measures is 24 × 24 sq. units. JKLM is a small square whose measure is 144 sq. units and this is fourth part of the square made by ABCD which is also equal to EBGF.

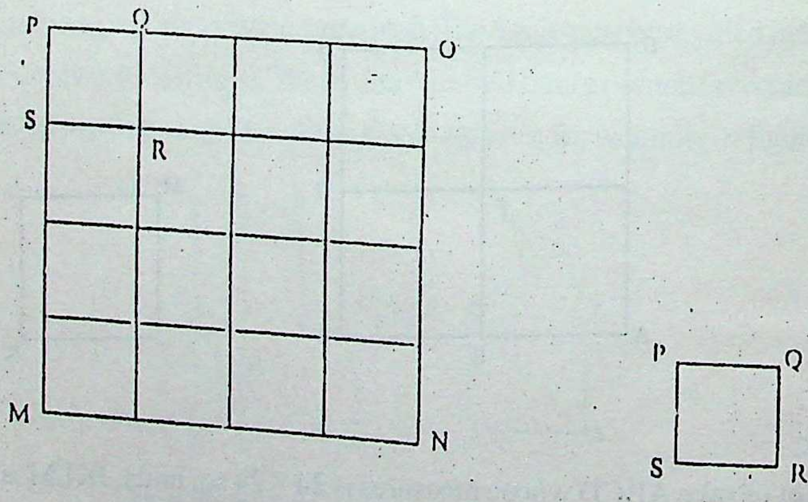
तृतीयेन नवमोंऽशः ॥ 3/9 ॥

The square formed by one third the length of given measure is ninth part of total area of the given measure of unit. e.g. MP is given : 3 unit of measure, its third part is taken and square EFGH is completed. Now MNOP square is being completed and we see that PQRS is ninth part of MNOP. Area of this part PQRS is same as that of EFGH.



चतुर्थेन षोडशी कला ॥ 3/10 ॥

Area of the sqare which is one fourth of the length of a measure is sixteenth part of the square formed by full length of a measure. How can this area be separately deducted (nirhavasa).
[Consult Kss 3/1].



एष निर्ह्रासस्तस्य पुरस्तादुक्तं शास्त्रम् ॥ 3/11 ॥

To deduct fourth part from 2 units of measure, ninth part from 3 units of measure, sixteenth part from 4 units of measure of side and so on, is already explained.

यावत्प्रमाणा रज्जुर्भवतीति विवृद्धे ह्रासाभवति ॥ 3/12 ॥

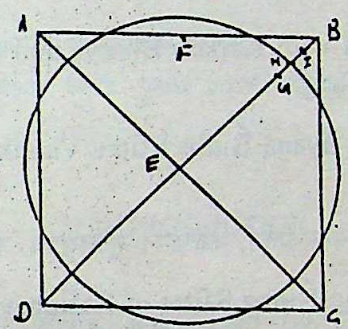
Deduction and entargement are made according to unit length of the chord. If lenght of chord is nine unit then its third part can be deducted. Similarly fourth part from sixteen units.

TRANSFORMATION OF SQUARE INTO CIRCLE & VICE-VERSA

चतुरस्रं मण्डलं चिकीर्षन्मध्यादसे निपात्य पार्श्वतः परिलिख्य तत्र

यदतिरिक्तं भवति तस्य-तृतीयेन सह मण्डलं परिलिखेत्स समाधिः ॥ 3/13 ॥

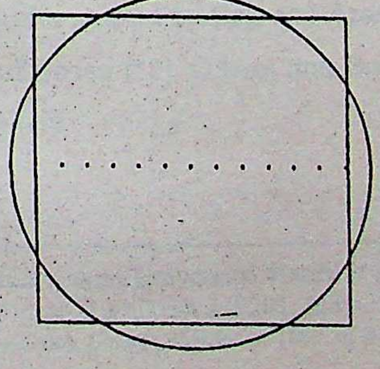
Transform ABCD square into a circle. Draw diagonal AC,BD from angle A and angle D. E is their meetting point, cut half of AB at F cut off on arc EG = FB from E. Divide GB into three parts GH. HI, IB. Now leave two parts and with centre EH draw a circle whose area will be equal to that square (for details see 2,p.65).



मण्डलं चतुरस्रं चिकीर्षन् विष्कम्भं पंचदशभागान्

कृत्वा द्रवुद्धरेच्छेषः करणी ॥ 3/14॥

To convert a circle into square divide diameter into 15 parts.



Give up two parts out of them. With the help of remaining chord which is of 13 parts, draw a square. Now area of a square will be equal to that of the given circle (fordetails see 2, P 65).

After discussing the varous ṣutras composed by Kātyāyana regarding the conversion of area it is concluded that in vedic age, there were various figures of different dimensions. It is also mentioned that in those days the device of dealing with the mathematical problems were very much in practice.

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New Coincidences and Fixed Points of Reciprocally Continuous and Compatible Hybrid Maps

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ABSTRACT

It is proved that a pair of reciprocally continuous and non-vacuously compatible single-valued and multivalued maps on metric spaces possesses a coincidence. This result is applied to obtain new general coincidence and fixed point theorems for single-valued and multivalued maps in metric spaces under tight minimal conditions.

Mathematics subject classifications (2000): 54H25, 47H10

Keywords and phrases: Coincidence point, fixed point, hybrid maps, reciprocally continuous, compatible maps.

INTRODUCTION

The notion of compatible maps, introduced by G. Jungck [7] (also called asymptotically commuting maps, cf. [24]) has proved useful for generalizing known results in the context of metric fixed point theory for continuous single-valued and multivalued maps (see, for instance, [1] - [2], [4], [7], [8], [11], [16], and [22]). Recently, reciprocal continuity for a pair of (discontinuous) single-valued maps (*resp.* hybrid pair of maps) has been introduced in [16] (*resp.* [21]).

In this paper we obtain coincidence and fixed point theorems for a hybrid quadruple of maps on a metric space satisfying very general contractive type conditions which include several general conditions studied by Beg and Azam [1], Ćirić [3], Jungck [8], Kaneko [10], [11], Kaneko and Sessa [12], Lohani and Badshah [13], Rhoades *et al.* [17], Tan and Minh [23] and others.

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PRELIMINARIES

Consistent with [1], [14], [15] and [20], we will use the following notations, where (X, d) is a metric space and $CL(X)$ is a collection of all nonempty closed subsets of X. Further, $d(A, B)$ denotes the ordinary distance between nonempty subsets of A and B of X, while $d(x, B)$ stands for $d(A, B)$ when $A = \{x\}$. The distance function H is called the generalized Hausdorff metric for $CL(X)$ induced by the metric d of X.

Definition 2.1 (Singh and Mishra [21]). The maps $T: X \to CL(X)$ and $f: X \to X$ are reciprocally continuous (r. c.) on X (*resp.* at $t \in X$) if and only if $fTx \in CL(X)$ for each $x \in X$ (*resp.* $fTt \in CL(X)$) and $\lim_{n\to\infty} fTx_n = fM$, $\lim_{n\to\infty} Tfx_n = Tt$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Tx_n = M \in CL(X),\ \lim_{n\to\infty} fx_n = t \in M.$$

For self-maps $f, g: X \to X$, this definition due to Pant [16] reads: f and g are r.c if and only if $\lim_{n\to\infty} gfx_n = gt$ and $\lim_{n\to\infty} fgx_n = ft$ whenever $\{x_n\} \subset X$ is such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = t \in X$. Clearly any continuous pair is reciprocally continuous but the converse is not true (Singh and Mishra [21, Example 2.3]). We remark that reciprocally continuous maps may be discontinuous even at their common fixed points (see [21]). For continuity of multivalued maps at their fixed and common fixed points, refer to [5].

The following definition is due to Kaneko and Sessa [12] and Beg and Azam [1] when $T: X \to CB(X)$.

Definition 2.2 (Singh and Mishra [21]). The maps $T: X \to CL(X)$ and $f: X \to X$ are compatible if and only if $fTx \in CL(X)$ for each $x \in X$ and $\lim_{n\to\infty} H(Tfx_n, fTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Tx_n, = M \in CL(X) \text{ and } \lim_{n\to\infty} fx_n = t \in M.$$

Evidently commuting maps T, f (i.e., when $fTx = Tfx$, $x \in X$) are weakly commuting (i.e., $H(Tfx, fTx) \leq d(fx, Tx)$, $x \in X$, [10] and [19]), weakly commuting maps are compatible, and compatible T, f are weakly compatible (i.e., when $fTx = Tfx$, whenever $fx \in Tx$, see Jungck and Rhoades [9]), but the reverse implication is not true. For excellent discussion on the role of weak compatibility in fixed point considerations, refer to Jungck and Rhoades [*op. cit.*].

Definition 2.3 (Itoh and Takahashi [10]). The maps $T: X \to CL(X)$ and $f: X \to X$ are (IT)-commuting at a point $v \in X$ if $fTv \subset Tfv$ (cf. [21, page 629]).

Remark 2.4.

(i) (IT)-commutativity of a hybrid pair (T, f) at a point v is more general than its compatibility (cf. [21, Example 1]) and weak compatibility at the point v.

(ii) Continuous and commuting pair of hybrid maps (T, f) of a complete metric space need not have a coincidence; for example $Tx = \{1 + x\}$ and $fx = x$, $x \in [0, \infty)$, (see [21]).

(iii) The nonvacuous compatibility of T and f implies the existence of at least a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Tx_n = M \in CL(X)$ and $\lim_{n\to\infty} fx_n = t \in X$). For details see [21].

MAIN RESULTS

The following is the main result. In all that follows, $C(S, A)$ stands for the collection of coincidence points of S and A, that is, $C(S, A) = \{v;\ Sv \in Av\}$.

Theorem 3.1. Let (X, d) be a metric space and $A_i: X \to CL(X)$, i = 1, 2, ... and $S, T: X \to X$ such that

(i) $A_1X \subset TX$ and the pair (A, S) is reciprocally continuous and nonvacuously compatible,

(ii) $H(A_1x, A_iy) < (m_{1i}(x, y))$, $i > 1$ and $x, y \in X$ when $m_{1i}(x, y) > 0$,

where $(m_{1i}(x, y)) = \max\{d(Sx, A_1x), d(Ty, A_iy), d(Sx, A_iy), d(Ty, A_1x), d(Sx, Ty)\}$.

Then $C(S, A_i)$ and (T, A_i) are nonempty. Further,

(I) S and A_1 have a common fixed point Su, provided $SSu = Su$ for some $u \in C(S, A_1)$;

(II) T and A_i have a common fixed point Tw, provided $TTw = Tw$ and T, A_i are (IT)-commuting at $w \in C(T, A_i)$;

(III) S, T, A and A_i have a common fixed point, provided (i) and (ii) both are true.

Proof. Since A_1 and S are nonvacuously compatible, there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} A_1x_n = M \in CL(X)$ and $\lim_{n\to\infty} Sx_n = u \in M$ and $\lim_{n\to\infty} H(A_1Sx_n, SA_1x_n) = 0$. This, in view of r. c. of A_1 and S, yields $H(A_1u, SM) = 0$, and $A_1u = SM$. Now $u \in M$ implies $Su \in A_1u$. Therefore $C(S, A_1)$ is nonempty. Since $A_1X \subset TX$, there exists a point $w \in X$ such that $Su = Tw \in A_1u$. If $A_iw \neq Tw$, then

$$d(Tw, A_iw) \leq H(A_1u, A_iw) < \max \{d(Su, A_1u), d(Tw, A_iw), d(Tw, A_1u), d(Su, A_iw), d(Tw, Su)\}$$
$$= d(Tw, A_iw),$$

a contradiction. This implies that $Tw \in A_iw$, and $C(T, A_i)$ is nonempty.

Further, $Su = SSu$ and the (IT)-commutativity of A and S at $u \in C(S, A)$ imply that $Su \in SA_1u = A_1Su$. So Su is a common fixed point of A and S.

The proof of (II) is similar, and (III) is immediate.

Corollary 3.2. Let (X, d) be a metric space, $A, B: X \to CL(X)$ and $S, T: X \to X$ such that

(iii) $AX \subset TX$ and the pair (S, A) is reciprocally continuous and nonvacuously compatible;

(iv) $H(Ax, By) \leq q(M(x, y))$ for each x, y in X and $q \in (0, 1)$, where
$M(x, y) = \max \{d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Sx), d(Sx, Ty)\}$.

Then $C(S, A)$ and $C(T, B)$ are nonempty. Further,

(Ia) S and A have a common fixed point Su, provided $SSu = Su$ for some $u \in (S, A)$;

(IIa) T and B have a common fixed point Tw, provided $TTw = Tw$ and T, B are (IT)-commuting at $w \in C(T, B)$;

(IIIa) S, T, A and B have a common fixed point, provided (Ia) and (IIa) both are true.

Proof. It may be completed following the proof of Theorem 3.1.

We remark that Theorem 3.1 with all maps single-valued presents a generalization of the main result of Pant [16, page 148]. Further, the above results extend some of the results of Singh and Kumar [18] to a quadruplet of hybrid maps.

Theorem 3.3. Corollary 3.2 with (iv) replaced by following :

(v) H(Ax, By) < $L(x, y)$ for all x, y in X when $L(x, y) > 0$, where

$$L(x, y) = \frac{\alpha\, d(Ty, By)[1 + d(Sx, Ax)]}{1 + d(Sx, Ty)} + \beta\, [\, d(Ty, By) + (Sx, Ax)]$$

$$+ \gamma[d(Sx, By) + d(Ty, Ax)] + \delta\, d(Sx, Ty),$$

where α, β, γ, $\delta \geq 0$ and $\alpha + \beta + 2\gamma + \delta < 1$.

Proof. It may be completed following the proof of Theorem 3.1.

Remark 3.4. Theorem 3.3 generalizes a result of Lohani and Badshah [13] and others.

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COINCIDENCE AND FIXED POINT THEOREMS FOR HYBRID CONTRACTIONS

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ABSTRACT

The purpose of this paper is to obtain coincidence and fixed point theorems for a pair of hybrid contracting maps.

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Key words and phrases: Coincidence point, fixed point, hybrid contracting maps, hybrid contraction.

1. INTRODUCTION

Consistent with [1], [4], [11], [16], [18], [20] and [23], we will use the following notations, where (X, d) is a metric space. Let $CL(X)$ denote the collection of all nonempty closed subsets of X. The distance function H on $CL(X)$ is called the generalized Hausdorff metric induced by the metric d of X. Further, let $d(A, B)$ denote the ordinary distance between nonempty subsets A and B of X, while $d(A, x)$ stands for $d(A, B)$ when B is the singleton $\{x\}$.

Let $T : X \to CL(X)$ be a multivalued map and $f : X \to X$ a single

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-valued map. Consider the following condition essentially due to Singh and Kulshrestha [21].

$$H(Tx, Ty) \leq q.\max \{d(fx, fy), d(fx, Tx), (fy, Ty), [d(fx, Ty) + d(fy, Tx)]/2\} \qquad \text{(SK)}$$

for all $x, y \in X$, where $q \in (0, 1)$.

We remark that (SK) with f = the identity map on X is Ćirić's generalized multivalued contraction (Ćirić [4]), which in turn includes the well-known Nadler multivalued contraction (Nadler, Jr. [16], see also [1], [4], [9]-[13], [15], [23] and [25]).

The study of hybrid contracting maps involving single-valued and multivalued maps on metric spaces was initiated independently by Bhaskaran and Subrahmanyam [2], Hadžić [5], Kaneko [9], Kubiak [15], Naimpally [17] *et al*. and Singh and Kulshrestha [21]. (Here, according to Singh and Mishra [22], "hybrid contracting maps" means a pair of hybrid contraction or nonexpansive or contractive). Indeed, Singh and Kulshrestha [*op. cit.*] (see also [12], [19] and [22]) showed that T and f satisfying $T(X) \subseteq f(X)$ and (SK) have a coincidence, that is, there exists a point $z \in X$ such that $fz \in Tz$ when $f(X)$ is a complete subspace of X. This result is obviously true when, instead of $f(X)$, $T(X)$ is a complete subspace of X . For an immediate excellent generalization of this result one may refer to Rhoades *et al*. [18].

In all that follows, Y is an arbitrary nonempty set and (X, d) a metric space. Following Liu *et al.* [14], Singh and Mishra [23] and Tan *et al.* [26], we shall consider the following condition for $f : Y \to X$ and $T : Y \to CL(X)$.

$$H^2(Tx, Ty) \leq q.m(x, y), \tag{1.1}$$

where $q \in (0, 1)$ and

$$\begin{aligned} m(x,\ y) := \max\ \{&d^2(fx, fy), d(fx, fy).d(fx, Tx), d(fx, fy).d(fy, Ty),\\ &d(fx, fy).[d(fx, Ty) + d(fy, Tx)]/2, d(fx, Tx).d(fy, Ty),\\ &d(fx, Tx).[d(fx, Ty) + d(fy, Tx)]/2, d(fy, Ty).[d(fx, Ty)\\ &+ d(fy, Tx)]/2, d(fx, Ty).d(fy, Tx)\}. \end{aligned}$$

Our main result is under the condition (1.1) (cf. Theorem 3). Following Rhoades *et al.* [18], we present generalized versions of this result as well.

2. RESULTS

DEFINITION 1. (Itoh and Takahashi [6], see also Singh and Mishra [24, p. 488]). Let Y be a nonempty set, $f : Y \to Y$ and $T : Y \to 2^Y$, the collection of all nonempty subsets of Y. Then the hybrid pair (T, f) is IT-commuting at $x \in Y$ if $fTx \subseteq Tfx$ for each $x \in Y$. (This formulation in [25] is correct with the interchange of symbols for single-valued and mutivalued maps).

We shall need the following result, which is a minor variant of a lemma due to Ćirić [4].

LEMMA 2. Let A, $B \in CL(X)$. Then for an $x \in A$ and for some q and k in (0, 1), there exists a $y \in B$ such that

$$d^2(x, y) \leq q^{-k} H^2(A, B).$$

THEOREM 3. Let Y be an arbitrary nonempty set and (X, d) a metric space. Let $T : Y \to CL(X)$ and $f : Y \to X$ be such that $T(Y) \subseteq f(Y)$ and (1.1) holds for all $x, y \in Y$. If $T(Y)$ or $f(Y)$ is a complete subspace of X then T and f have a coincidence. Indeed, for any $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that

(I) $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots$;

(II) $\{fx_n\}$ converges to fz for some $z \in Y$, and $fz \in Tz$, that is, T and f have a coincidence at z; and

(III) $d(fx_n, fz) \leq [\beta^n/(1-\beta)].d(fx_0, fx_1)$, where $\beta = q^{1-k}[1 + \sqrt{(1 + 8q^{-1+k})}]/4$ for some $k \in (0, 1)$.

Further, if $Y = X$ and the pair (T, f) is IT-commuting at z then T and f have a common fixed point provided that $ffz = fz$.

PROOF. Pick a point x_0 in Y. Let k be a positive number such that $k < 1$. Following Singh and Kulshrestha [21] (see also [12]), we construct sequences $\{x_n\} \subseteq Y$ and $\{fx_n\} \subseteq X$ in the following manner.

Since $T(Y) \subseteq f(Y)$, we may choose a point $x_1 \in Y$ such that $fx_1 \in Tx_0$. If $Tx_0 = Tx_1$ then $x_1 = z$ is a coincidence point of T and f, and we are done. So assume that $Tx_0 \neq Tx_1$ and choose $x_2 \in Y$ such that $fx_2 \in Tx_1$ and, by Lemma 2,

$$d^2(fx_1, fx_2) \leq q^{-k} H^2(Tx_0, Tx_1).$$

If $Tx_1 = Tx_2$, then x_2 becomes a coincidence point of T and f. If not, continue the process. In general, if $Tx_n \neq Tx_{n+1}$, we choose $fx_{n+2} \in Tx_{n+1}$ such that

$$d^2(fx_{n+1}, fx_{n+2}) \leq q^{-k} H^2(Tx_n, Tx_{n+1}).$$

Then by (1.1),

$$\begin{aligned}
d^2(fx_{n+1}, fx_{n+2}) &\leq q^{-k}H^2(Tx_n, Tx_{n+1})\\
&\leq q^{1-k}\max \{d^2(fx_n, fx_{n+1}), d(fx_n, fx_{n+1}).d(fx_n, Tx_n), d(fx_n, fx_{n+1}).d(fx_{n+1}, Tx_{n+1}),\\
&\quad d(fx_n, fx_{n+1}).[d(fx_n, Tx_{n+1}) + d(fx_{n+1}, Tx_n)]/2, \ d(fx_n, Tx_n).d(fx_{n+1}, Tx_{n+1}),\\
&\quad d(fx_n, Tx_n).[d(fx_n, Tx_{n+1}) + d(fx_{n+1}, Tx_n)]/2, d(fx_{n+1}, Tx_{n+1}).[d(fx_n, Tx_{n+1})\\
&\quad + d(fx_{n+1}, Tx_n)]/2, \ d(fx_n, Tx_{n+1}. d(fx_{n+1}, Tx_n)\}.
\end{aligned}$$

For the sake of simplicity, let $d_n := d(fx_n, fx_{n+1})$.

Then the above inequality, after simplification, reduces to

$$d_{n+1}^2 \leq q^{1-k}. \max \{d_n^2, d_nd_{n+1}, d_n[d_n + d_{n+1}]/2, d_{n+1}[d_n + d_{n+1}]/2\}. \tag{3.1}$$

We remark that in the construction of sequences $\{x_n\}$ and $\{fx_n\}$, x_n (for each n) is not a coincidence point of T and f. This together with $Tx_n \neq Tx_{n+1}$ means that $fx_n \neq fx_{n+1}$. Indeed, if at any stage $fx_n = fx_{n+1}$ then

$fx_n \in Tx_n$, and x_n is a coincidence point of T and f. Therefore, according to our construction of the sequences, $d_n \neq 0$. Hence the inequality (3.1) implies one of the following:

$$d_{n+1}^2 \leq \lambda\, d_{n+1}^2, \qquad \text{where } \lambda = q^{1-k};$$

$$d_{n+1} \leq \lambda\, d_n;$$

$$d_{n+1} \leq [\lambda/4 + \sqrt{(\lambda^2/16 + \lambda/2)}]d_n;$$

$$d_{n+1} \leq [\lambda/(2 - \lambda)]d_n.$$

Consequently,

$$d_{n+1} \leq \max\,\{\sqrt{\lambda}, \lambda, [\lambda/4 + \sqrt{(\lambda^2/16 + \lambda/2)}], [\lambda/(2 - \lambda)]\}\ d_n.$$

This gives $d_{n+1} \leq \beta d_n$, when $\beta = [\lambda/4 + \sqrt{(\lambda^2/16 + \lambda/2)}]$. Notice that $0 < \beta < 1$. Hence $\{fx_n\}$ is a Cauchy sequence.

Now let $f(Y)$ be a complete subspace of Y. Then the sequence $\{fx_n\}$ has a limit $f(Y)$. Call it b. Hence there exists a point $z \in Y$ such that $fz = b$. Since $d(fz, Tz) \leq d(fz, fx_{n+1}) + H(Tx_n, Tz)$, applying (1.1) to the last term of this inequality and making $n \to \infty$, the inequality gives $d(fz,\ Tz) \leq \sqrt{q}.d(fz,\ Tz)$, and $fz \in Tz$, since $\sqrt{q} < 1$ and Tz is closed. This argument applies to the case when $T(Y)$ is a complete subspace of Y, since $T(Y) \subseteq f(Y)$.

This proves (I) and (II). To see (III), let $m > n$. Then,

$$d(fx_n, fx_m) \leq d(fx_n,\ fx_{n+1}) + \ldots + d(fx_{m+1}, fx_m)$$

$$\leq (\beta^n + \beta^{n+1} + \ldots + \beta^{m-1}).\ d(fx_0,\ fx_1)$$

$< \beta^n/(1 - \beta).d(fx_0, fx_1)$.

Making $m \to \infty$, we get (III).

Let $\psi = \{\phi: R_+ \to R_+ | \phi$ is upper semicontinuous and nondecreasing$\}$. Then, in view of the proof of theorem 3 (above) and Rhoades *et al*. [18, Theorem 1], we have two generalizations of Theorem 3. We shall need the following definitions.

The following three definitions are due to Rhoades *et al*. [18] when $Y = X$.

DEFINITION 4. Let Y be an arbitrary nonempty set and (X, d) a metric space. If for a point $x_0 \in Y$ there exists a sequence $\{x_n\}$ in Y such that $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots$, then $O(T, f; x_0) := \{fx_n : n = 1, 2, \ldots\}$ is the orbit for (T, f) at x_0. Further $O(T, f; x_0)$ is called a regular orbit for (T, f) if for each n,

$$d(fx_{n+1}, fx_{n+2}) \leq H(Tx_n, Tx_{n+1}).$$

We remark that the symbol for the orbit $O(T, f; x_0)$ is sometimes used for a sequence.

DEFINITION 5. A subspace $T(Y)$ or $f(Y)$ is called (T, f)-orbitally complete if every Cauchy sequence of the form $\{fx_{n_i} : fx_{n_i} \in Tx_{n_{i-1}}\}$ converges in X, where $\{x_n\}$ is a sequence in Y.

DEFINITION 6. If, for a point $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that the sequence $O(T, f; x_0)$ converges in X then X is called (T, f)-orbitally complete with respect to x_0 or simply $(T, f; x_0)$-orbitally complete. Further, an orbit $O(T, f; x_0)$ is said to be asymptotically regular if $\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0$, where fx_n and fx_{n+1} are from $O(T, f; x_0)$.

THEOREM 7. Let Y be a nonempty set and (X, d) a metric space, $T : Y \to CL(X)$ and $f : Y \to X$ such that $T(Y) \subseteq f(Y)$ and the following hold:

$$H^2(Tx, Ty) \le \phi(m(x, y)) \text{ for all } x, y \in Y; \tag{7.1}$$

$$\phi(t) < qt \qquad \text{for each } t > 0, \tag{7.2}$$

for some $q \in (0, 1)$ and $\phi \in \psi$.

If $T(Y)$ or $f(Y)$ is a complete subspace of X then T and f have a coincidence point z (say). Further if $Y = X$ and the pair (T, f) are IT-commuting at z then T and f have a fixed point provided that $ffz = fz$.

THEOREM 8. Let Y be a nonempty set, (X, d) a metric space, $T : Y \to CL(X)$ and $f : Y \to Y$ such that

$$H^2(Tx, Ty) \le \phi(M(x, y)), \tag{8.1}$$

where

$$\begin{aligned} M(x, y) := q.\max \{ & d^2(fx, fy), d(fx, fy).d(fx, Tx), d(fx, fy).d(fy, Ty), \\ & d(fx, fy).d(fx, Ty), d(fx, fy).d(fy, Tx), d(fx, Tx).d(fy, Ty), \\ & d(fx, Tx).d(fx, Ty), d(fx, Tx).d(fy, Tx), d(fy, Ty).d(fx, Ty), \\ & d(fy, Ty).d(fy, Tx), d(fx, Ty).d(fy, Tx)\}. \end{aligned}$$

for all $x, y \in Y$.

$$\phi(t) < t \text{ for each } t > 0, \quad \phi \in \psi. \tag{8.2}$$

If, for some $x_0 \in Y$, there exists a sequence $\{x_n\} \subseteq X$ such that the orbit $O(T, f; x_0)$ is regular and asymptotically regular, and $T(X)$ or $f(X)$ is $O(T, f; x_0)$-orbitally complete, then T and f have a coincidence. Further, if $Y = X$ and the pair (T, f) is IT-commuting just at a coincidence point z (say) and $ffz = fz$, then fz is a common fixed point of T and f.

We remark that the completeness requirement in Theorem 3 may evidently be replaced by orbital completeness. Further, for the existence of a common fixed point of T and f, the requirement $ffz = fz$ can not be dropped from the above theorems (see, for instance, [17], [21], [22, page 194-195] and [24]). This means that, under either of the above theorems with $Y = X$, a coincidence point of T and f need not be their fixed point.

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

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मैं ए0के0 चोपड़ा, कुलसचिव, गुरुकुल कांगड़ी विश्वविद्यालय, हरिद्वार, घोषित करता हूँ कि उपरिलिखित तथ्य मेरी जानकारी के अनुसार सही हैं।

हस्ताक्षर

डॉ0 ए0के0 चोपड़ा

कुलसचिव